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Symmetry Properties of the Approximations of Multidimensional Generalized Van Der Pol Equations

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Abstract

We calculate symmetries of the nonlinear n-dimensional wave equation $\Box_n u + m^2 u - \varepsilon f(u)\left(\sum\limits_{\nu=0}^{n-1}\lambda_\nu \partial u/\partial x_\nu\right) = 0$ and its approximations. In particular, we study the conformal invariance of the equations. Conditional symmetries of the approximate system are also considered whereby approximate solutions are constructed.

1 Introduction

Recently we studied approximate symmetries for a Landau-Ginzburg equation [1] and a multidimensional nonlinear heat equation [2]. Within this approach one can obtain approximate solutions for multidimensional partial differential equations with a small parameter [3–5]. By approximate symmetries we mean the exact symmetries of an approximate system of the original partial differential equation. Different representations of the solutions can be considered. In [2] we studied first order approximations of a nonlinear multidimensional heat equation under the representations $u = u_0 + \varepsilon u_1$ and $u = u_0 + \varepsilon g(u_0)$.

For the second representation a conditional equation for u_0 appeared, whereby we obtained compatible solutions by studying conditional symmetries for the equation in u_0.

In this paper we study the nonlinear multidimensional wave equation

$$\Box_n u + m^2 u - \varepsilon f(u) \left(\sum_{\nu=0}^{n-1} \lambda_\nu \frac{\partial u}{\partial x_\nu} \right) = 0, \tag{1}$$

where f is an arbitrary differentiable fuction, ε is a small real positive parameter, and m^2, λ_ν are real constants. Here

$$\Box_n \equiv \frac{\partial^2}{\partial x_0^2} - \sum_{j=1}^{n-1} \frac{\partial^2}{\partial x_j^2}.$$

For the case

$$f(u) = 1 - u^2, \tag{2}$$

with $\lambda_j = 0$ ($j = 1, \ldots, n-1$) and $n = 4$, equation (1) was proposed and named the multidimensional Van der Pol equation [6]. Note that for $n = 1$ the classical Van der Pol equation is obtained. Based on the conformal symmetry properties of (1) we connect the dimension of the equation to the nonlinear function f. For the n-dimensional case the function f, for conformal-type invariance, is given by

$$f(u) = c_1 + c_2 u^{4/(n-2)} \tag{3}$$

for first approximation (see theorem 6). We named (1) with the nonlinear function (3) the multidimensional generalized Van der Pol equation. Based on the symmetries of (1) and its approximations one can construct approximate solutions that satisfy (1) in the order of approximation. Q-conditional symmetries [7, 5, 8] whereby non-Lie ansätze can be constructed are also considered for an approximate system of (1).

2 Exact Symmetry Reductions

In this section we give the maximal Lie symmetry algebra of (1), where f is arbitrary. We also study (1) with respect to conformal invariance, i.e., the conformal symmetry generators

$$K_{0u} = 2x_0 \sum_{i=1}^{n-1} x_i \frac{\partial}{\partial x_i} + \sum_{\mu=0}^{n-1} x_\mu^2 \frac{\partial}{\partial x_0} - (n-2) x_0 u \frac{\partial}{\partial u}, \tag{4}$$

$$K_{lu} = 2x_l \sum_{\mu=0}^{n-1} x_\mu \frac{\partial}{\partial x_\mu} - \left(\sum_{k=1}^{n-1} x_k^2 - x_0^2 \right) \frac{\partial}{\partial x_l} - (n-2) x_l u \frac{\partial}{\partial u}, \tag{5}$$

where $l = 1, \ldots, (n-1)$. Using these symmetry properties the wave equations can be reduced to classical Van der Pol type equations or, sometimes even linear equations. Note that the nonlinear reduced ordinary differential equations can be studied by the use of the Krylov–Bogolubov method [9] in order to obtain approximate solutions for the nonlinear wave equations. The methods of performing such calculations are well known and will not be considered here. We are mainly interested in the symmetry properties of the equations.

Using standard Lie methods [10–13, 5] we can prove the following

Theorem 1 *The basis elements of the maximal Lie symmetry algebra of (1) is given by the following Lie symmetry generators*

$$< T_\nu,\ \tilde{R}_{lj} >, \tag{6}$$

where

$$\begin{aligned}
&T_\nu = \frac{\partial}{\partial x_\nu}, \qquad \tilde{R}_{lj} = \lambda_j R_{l1} + \lambda_l R_{1j} + \lambda_1 R_{jl},\\
&R_{ab} = x_a \frac{\partial}{\partial x_b} - x_b \frac{\partial}{\partial x_a},\\
&\nu = 0, \ldots, n-1, \quad l = 2, 3, \ldots, n-2, \quad j = l+1, \ldots, n-1.
\end{aligned}$$

If $\lambda_0 = 1$ and $\lambda_j = 0$ $(j = 1, \ldots, n-1)$ (1) admits the Lie symmetries

$$< T_\nu,\ R_{ij} >,$$

where $\nu = 0, \ldots, n-1$ and $i \neq j = 1, \ldots, n-1$.

For the function f given in (2), symmetry reductions to classical Van der Pol equations were performed in [5]. We are not concerned with such reductions in this paper. Let us, however, mention that (3), with all λ's different from zero can easily be reduced to a linear equation with the use of the translation symmetries. For example. Let $\lambda_j = 1$ $(j = 0, 1, \ldots, n-1)$, and consider the ansatz

$$u(x_0, \ldots, x_{n-1}) = \varphi(\omega_1, \omega_2),$$

where

$$\begin{aligned}
\omega_1 &= \sum_{j=1}^{n-1} x_j - (n-1)x_0,\\
\omega_2 &= \sum_{j=1}^{n-1} x_j^2 - 2\sum_{j=1}^{n-1} x_j x_0 + (n-1)x_0^2,
\end{aligned}$$

are invariant functions of the translation symmetries. Since ω_1 and ω_2 are also first integrals of the equation

$$\frac{\partial u}{\partial x_0} + \cdots + \frac{\partial u}{\partial x_{n-1}} = 0,$$

(1) reduces to the linear equation

$$(n-1)(n-2)\frac{\partial^2\varphi}{\partial\omega_1^2}+4(\omega_1^2-\omega_2)\frac{\partial^2\varphi}{\partial\omega_2^2}-4n\omega_1\frac{\partial^2\varphi}{\partial\omega_1\partial\omega_2}+m^2\varphi=0.$$

Let us now turn to the question of conformal invariance of the nonlinear wave equation (1). We state the following

Theorem 2 *The nonlinear wave equation*

$$\Box_n u+m^2u-\varepsilon f(u)\left(\sum_{\nu=0}^{n-1}\lambda_\nu\frac{\partial u}{\partial x_\nu}\right)=0$$

admits a combination of conformal symmetry generators K_{0u} and K_{lu} (given by (4) and (5), respectively) if and only if $m^2=0$ and

$$f(u)=c_1u^{4/(n-2)} \tag{7}$$

(with $n\neq 2$, $c_1\in\mathcal{R}$) for the following cases
i) $\lambda_0=\pm\lambda_l\neq 0$, $0<l\leq n-1$ and $\lambda_j=0$, $j=1,\ldots,\widehat{l},\ldots,n-1$. The circumflex indicates omission. The equation then admits the following combination of conformal generators:

$$K_{0u}\mp K_{lu}.$$

ii) $\lambda_0^2-\sum_{j=1}^{n-1}\lambda_j^2=0$, $\lambda_0\neq 0$. The equation then admits the combination

$$\lambda_0K_{0u}-\sum_{j=1}^{n-1}\lambda_jK_{ju}.$$

The proof of this theorem is similar to that of theorem 6 which is performed in the Appendix.

REMARK: *Function (7) can be given in the more general form*

$$f(u)=(c_1u+c_2)^{4/(n-2)},$$

whereby the infinitesimal function η of $\partial/\partial u$ in K_{0u} and K_{lu} depend on c_1 and c_2. We considered η as given in (4) and (5), only.

3 Lie Symmetries of the Approximate system

Let us introduce the following representation for u:

$$u=\sum_{j=0}^{k}\varepsilon^ju_j. \tag{8}$$

At first we consider the equation

$$\Box_n u + m^2 u - \varepsilon(1-u^2)\left(\sum_{\nu=0}^{n-1} \lambda_\nu \frac{\partial u}{\partial x_\nu}\right) = 0. \tag{9}$$

By representation (8) the k-order approximate system of (9) is given by the following $(k+1)$ partial differential equations:

$$\begin{aligned}
&\Box_n u_0 + m^2 u_0 = 0,\\
&\Box_n u_1 + m^2 u_1 + u_0^2\left(\sum_{\nu=0}^{n-1} \lambda_\nu \frac{\partial u_0}{\partial x_\nu}\right) - \sum_{\nu=0}^{n-1} \lambda_\nu \frac{\partial u_0}{\partial x_\nu} = 0,\\
&\Box_n u_2 + m^2 u_2 + u_0^2\left(\sum_{\nu=0}^{n-1} \lambda_\nu \frac{\partial u_1}{\partial x_\nu}\right) - \sum_{\nu=0}^{n-1} \lambda_\nu \frac{\partial u_1}{\partial x_\nu} + 2u_0u_1\left(\sum_{\nu=0}^{n-1} \lambda_\nu \frac{\partial u_0}{\partial x_\nu}\right) = 0,\\
&\Box_n u_j + m^2 u_j + \sum_{\nu=0}^{n-1} \sum_{0\le\mu\le(j-1)/2} \lambda_\nu u_\mu^2 \frac{\partial u_{j-(2\mu+1)}}{\partial x_\nu} - \sum_{\nu=0}^{n-1} \lambda_\nu \frac{\partial u_{j-1}}{\partial x_\nu} +\\
&\qquad 2\sum_{\nu=0}^{n-1} \sum_{0\le\mu\le(j-2)/2} \sum_{l=\mu+1}^{j-(\mu+1)} \lambda_\nu u_\mu u_{j-l} \frac{\partial u_{l-(\mu+1)}}{\partial x_\nu} = 0,
\end{aligned} \tag{10}$$

where $j = 3, 4, \ldots, k$. System (10) admits the infinite symmetry generator

$$Z_\infty = h(x_0, \ldots, x_{n-1}) \frac{\partial}{\partial u_k}, \tag{11}$$

where h is a solution of $\Box_n h + m^2 h = 0$. This reflects the linearity of the k-th equation in the dependent variable u_k. We now study the Lie symmetries of the above system of k-partial differential equations, i.e, the k-order approximate symmetries of (9) $(k > 0)$. The proof of the following theorems can be performed by the use of the standard Lie method and mathematical induction. In order to study system (10) with respect to Lie symmetries we have to consider different cases.

Case 1 Consider $\lambda_0 = 1$, $\lambda_j = 0$ $(j = 1, \ldots, n-1)$, i.e., the equation

$$\Box_n u + m^2 u - \varepsilon(1-u^2)\frac{\partial u}{\partial x_0} = 0. \tag{12}$$

Theorem 3 *Equation (12) admits the following Lie symmetry generators under the representation (8):*
i) $m^2 \neq 0$:

$$< T_\nu,\ R_{ij},\ \Gamma_{0k},\ \Lambda^{(2)}_{x_0 k} >, \qquad \nu = 0, \ldots, n-1, \quad i \neq j = 1, \ldots, n-1.$$

ii) $m^2 = 0$*:*

$$< T_\nu,\ R_{ij},\ S_k,\ \Gamma_{0k},\ \Lambda^{(2)}_{x_0k} >, \qquad \nu = 0,\dots,n-1, \quad i \neq j = 1,\dots,n-1.$$

Here

$$\Gamma_{0k} = u_0 \frac{\partial}{\partial u_k}, \tag{13}$$

$$\Lambda^{(2)}_{x_0k} = (u_0x_0 - 3u_1)\frac{\partial}{\partial u_k} - u_0\frac{\partial}{\partial u_{k-1}}, \tag{14}$$

$$S_k = S_{u_k} + S_x$$

$$\text{with} \quad S_{u_k} = \sum_{j=1}^{k} j u_j \frac{\partial}{\partial u_j}, \quad S_x = \sum_{\mu=0}^{n-1} x_\mu \frac{\partial}{\partial x_\mu}. \tag{15}$$

Case 2 $\lambda_0 \neq 0,\ \lambda_l \neq 0,\ \lambda_j = 0,\ (j = 1,\dots,\widehat{l},\dots,n-1)$. This results in the equation

$$\Box_n u + m^2 u - \varepsilon(1-u^2)\left(\lambda_0\frac{\partial u}{\partial x_0} + \lambda_l \frac{\partial u}{\partial x_l}\right) = 0. \tag{16}$$

Theorem 4 *Under the representation (8) we obtain the following Lie symmetry generators for (16):*
i) $m^2 \neq 0$ *and* $\lambda_0 = -\lambda_l$*:*

$$< T_\nu,\ R_{ab},\ L_{0j} - R_{j\tilde{l}},\ L_{0i} - R_{\tilde{l}i},\ S_{u_k} - L_{0l},\ \Gamma_{0k},\ \Lambda^{(2)}_{x_0+x_lk} > .$$

ii) $m^2 \neq 0$ *and* $\lambda_0 = \lambda_l$*:*

$$< T_\nu,\ R_{ab},\ L_{0i} - R_{\tilde{l}i},\ S_{u_k} + L_{0l},\ \Gamma_{0k},\ \Lambda^{(2)}_{x_0-x_lk} > .$$

iii) $m^2 = 0$ *and* $\lambda_0 = -\lambda_l$*:*

$$< T_\nu,\ R_{ab},\ L_{0j} - R_{j\tilde{l}},\ L_{0i} - R_{\tilde{l}i},\ S_k,\ \ S_x - L_{0l},\ \Gamma_{0k},\ \Lambda^{(2)}_{x_0+x_lk} > .$$

iv) $m^2 = 0$ *and* $\lambda_0 = \lambda_l$*:*

$$< T_\nu,\ R_{ab},\ L_{0i} - R_{\tilde{l}i},\ S_k\ S_x + L_{0l},\ \Gamma_{0k},\ \Lambda^{(2)}_{x_0-x_lk} > .$$

Let us define the symbols used in theorem 4:

$$\Lambda^{(2)}_{x_0\pm x_lk} = (u_0(x_0 \pm x_l) - 3u_1)\frac{\partial}{\partial u_k} - u_0\frac{\partial}{\partial u_{k-1}} \equiv \Lambda^{(2)}_{x_0k} \pm u_0 x_l \frac{\partial}{\partial x_k}, \tag{17}$$

also

$$a \neq b = 1,\dots,\widehat{l},\dots,n-1, \quad 0 < l \leq n-1, \quad j = 1,\dots,\tilde{l}-1,$$
$$i = \tilde{l}+1,\dots,n-1, \quad 1 < \tilde{l} \leq n-1, \quad \nu = 0\dots,n-1.$$

Case 3 $\lambda_0 \neq 0$ and all λ's are allowed to be different from zero, i.e., equation (9).

Theorem 5 *Under the representation (8) we obtain the following Lie symmetry generators for (9):*

i) $m^2 \neq 0$*:*

$$< T_\nu,\ \left(\lambda_0 R_{(a+b)a} + \lambda_a L_{0(a+b)} - \lambda_{(a+b)} L_{0a}\right),$$
$$\left(\lambda_0 S_{u_k} - \sum_{j=1}^{n-1} \lambda_j L_{0j}\right),\ \Lambda^{(2)}_{x_0 x_1 \dots x_{n-1} k},\ \Gamma_{0k} >;$$

ii) $m^2 = 0$ *and*

$$\lambda_0^2 - \lambda_1^2 - \cdots - \lambda_{n-1}^2 \neq 0 :$$

$$< T_\nu,\ \left(\lambda_0 R_{(a+b)a} + \lambda_a L_{0(a+b)} - \lambda_{(a+b)} L_{0a}\right),\ S_k,\ \Lambda^{(2)}_{x_0 x_1 \dots x_{n-1} k},\ \Gamma_{0k} >;$$

iii) $m^2 = 0$ *and*

$$\lambda_0^2 - \lambda_1^2 - \cdots - \lambda_{n-1}^2 = 0 :$$

$$< T_\nu,\ \left(\lambda_0 R_{(a+b)a} + \lambda_a L_{0(a+b)} - \lambda_{(a+b)} L_{0a}\right),\ S_k,$$
$$\left(\lambda_0 S_x + \sum_{j=1}^{n-1} \lambda_j L_{0j}\right),\ \Lambda^{(2)}_{x_0 x_1 \dots x_{n-1} k},\ \Gamma_{0k} > .$$

Here the symbols are defined as follows:

$$\Lambda^{(2)}_{x_0 x_1 \dots x_{n-1} k} = \left\{ \left(\lambda_0 x_0 - \sum_{j=1}^{n-1} \lambda_j x_j \right) u_0 - 3u_1 \right\} \frac{\partial}{\partial u_k} - u_0 \frac{\partial}{\partial u_{k-1}}. \tag{18}$$

Also, $a = 1, \dots, n-2,\ b = 1, \dots, n-a-1$.

From the Lie symmetries given above similarity ansätze for system (10), in the form

$$u_\mu(x_0, \dots, x_{n-1}) = \sum_{j=0}^{\mu} f_{\mu j}(x_0, \dots, x_{n-1}) \varphi_j(\omega_1, \dots, \omega_{n-1}) + f_\mu(x_0, \dots, x_{n-1}),$$

can be constructed. Here the ω's are the new independent variables and the φ's the new dependent variables.

As an example, we consider (12) together with its 1-dimensional invariant subalgebra

$$Z = \sum_{\nu=0}^{n-1} T_\nu + \Lambda^{(2)}_{x_0 k}.$$

The similarity ansatz is obtained by solving the associated Lagrange system. The ansätze for first-, second-, and k-order ($k > 2$) approximations are essentially different:
For first approximation we obtain the ansatz:

$$\begin{aligned} u_0 &= \exp(-x_0)\varphi_0(\omega), \\ u_1 &= \frac{1}{2}\left(x_0 - \frac{1}{2}\right)\exp(-x_0)\varphi_0(\omega) + \exp(-3x_0)\varphi_1(\omega). \end{aligned} \tag{19}$$

For second approximation we have the similarity ansatz

$$\begin{aligned} u_0 &= \varphi_0(\omega), \\ u_1 &= -x_0\varphi_0(\omega) + \varphi_1(\omega), \\ u_2 &= 2x_0^2\varphi_0(\omega) - 3x_0\varphi_1(\omega) + \varphi_2(\omega). \end{aligned} \tag{20}$$

For k-approximation (with $k > 2$) we obtain

$$\begin{aligned} u_k &= \frac{x_0^2}{2}\varphi_0(\omega) - 3x_0\varphi_1(\omega) + \varphi_k(\omega), \\ u_{k-1} &= -x_0\varphi_0(\omega) + \varphi_{k-1}(\omega), \\ u_{k-i} &= \varphi_{k-i}(\omega), \qquad i = 2, 3, \dots, k. \end{aligned} \tag{21}$$

Here ω is given by

$$\omega = \sum_{j=1}^{n-1} x_j - (n-1)x_0.$$

The reduced equations can be written down for any order of approximation. In this case the reduced equations are linear driven oscillators. Let us write down the reduced equations for first, second, and third order approximation: By application of the ansatz (19) to system (10) (for (12)), with $k = 1$, we obtain

$$\begin{aligned} &(n-1)(n-2)\varphi_0'' + 2(n-1)\varphi_0' + (m^2+1)\varphi_0 = 0, \\ &(n-1)(n-2)\varphi_1'' + 6(n-1)\varphi_1' + (m^2-9)\varphi_1 = (n-1)\varphi_0^2\varphi_0' + \varphi_0^3. \end{aligned}$$

For $k = 2$ the ansatz (20) applies. It follows that

$$\begin{aligned} &(n-1)(n-2)\varphi_0'' + m^2\varphi_0 = 0, \\ &(n-1)(n-2)\varphi_1'' + m^2\varphi_1 = -3(n-1)\varphi_0' + (n-1)\varphi_0^2\varphi_0', \\ &(n-1)(n-2)\varphi_2'' + m^2\varphi_2 = -7(n-1)\varphi_1' + (n-1)\varphi_0^2\varphi_1' + \\ &\qquad\qquad 2(n-1)\varphi_0\varphi_1\varphi_0' - 5\varphi_0 + \varphi_0^3. \end{aligned}$$

For $k = 3$ the ansatz (21) applies. It follows that

$$\begin{aligned}
&(n-1)(n-2)\varphi_0'' + m^2\varphi_0 = 0,\\
&(n-1)(n-2)\varphi_1'' + m^2\varphi_1 = -(n-1)\varphi_0'(1-\varphi_0^2),\\
&(n-1)(n-2)\varphi_2'' + m^2\varphi_2 = -(n-1)\varphi_1'(1-\varphi_0^2) - 2(n-1)\varphi_0'(1-\varphi_0\varphi_1),\\
&(n-1)n-2)\varphi_3'' + m^2\varphi_3 = -(n-1)\varphi_2'(1-\varphi_0^2) - 2(n-1)\varphi_1'(3-\varphi_0\varphi_1) +\\
&\qquad\qquad\qquad\qquad\qquad (n-1)\varphi_0'(\varphi_1^2 + 2\varphi_0\varphi_2) - \varphi_0(2-\varphi_0^2).
\end{aligned}$$

Here $' \equiv d/d\omega$. All reductions which follow for higher approximations are obtained by applying ansatz (21). In this way any order of approximation for (12) can be obtained by solving the obove linear ordinary differential equations.

Let us now turn to the question of the conformal invariance of (1) under the representation (8), i.e., we study the conformal invariance for the approximate system of (1). We can state the following

Theorem 6 *The nonlinear wave equation*

$$\Box_n u + m^2 u - \varepsilon f(u)\left(\lambda_0 \frac{\partial u}{\partial x_0} + \lambda_{i_1}\frac{\partial u}{\partial x_{i_1}} + \cdots + \lambda_{i_r}\frac{\partial u}{\partial x_{i_r}}\right) = 0, \tag{22}$$

with $r, i_1, \ldots, i_r \in \{1, \ldots, n-1\}$, $r = i_1 + \cdots + i_r$, $n \neq 2$, and $\lambda_0 \neq 0$, admits the Lie symmetry generators given by the cases (i) to (iv) below under the the representation (8), if and only if $m^2 = 0$ and

$$f(u) = c_1 u^{4/(n-2)} + c_2 \qquad (c_1, c_2 \in \mathcal{R}).$$

i) $\lambda_0 = -\lambda_l$.
For first approximation (22) admits the following Lie symmetry generators:

$$< T_\nu,\ R_{ab},\ L_{0j} - R_{j\tilde{l}},\ L_{0i} - R_{\tilde{l}i},\ S_1,\ S_x - L_{0l},\ \Gamma_{01},\ \Lambda^{4/(n-2)}_{(x_0+x_l)1},\ \Upsilon_{(x_0+x_l)} >$$

For k-order approximation ($k \geq 2$) (22) admits the Lie symmetry generators:

$$< T_\nu,\ R_{ab},\ L_{0j} - R_{j\tilde{l}},\ L_{0i} - R_{\tilde{l}i},\ S_k,\ S_x - L_{0l},\ \Gamma_{0k},\ \Lambda^{4/(n-2)}_{(x_0+x_l)k} >$$

Here

$$\begin{aligned}
&a \neq b = 1, \ldots, \widehat{l}, \ldots, n-1, \quad 0 < l \leq n-1,\\
&j = 1, \ldots, \tilde{l} - 1, \quad i = \tilde{l} + 1, \ldots, n-1,\\
&1 < \tilde{l} \leq n-1, \quad \nu = 0 \ldots, n-1.
\end{aligned}$$

ii) $\lambda_0 = \lambda_l$.
For first approximation (22) admits the following Lie symmetry generators:

$$< T_\nu,\ R_{ab},\ L_{0i} - R_{\tilde{l}i},\ S_1,\ S_x - L_{0l},\ \Gamma_{01},\ \Lambda^{4/(n-2)}_{(x_0-x_l)1},\ \Upsilon_{(x_0-x_l)} > .$$

For k-order approximation ($k \geq 2$) (22) admits the Lie symmetry generators:

$$< T_\nu,\ R_{ab},\ L_{0i} - R_{\tilde{l}i},\ S_k,\ S_x + L_{0l},\ \Gamma_{0k},\ \Lambda^{4/(n-2)}_{(x_0-x_l)k} > .$$

Here

$$\begin{array}{l} a \neq b = 1, \ldots, \widehat{l}, \ldots, n-1, \quad 0 < l \leq n-1, \\ j = 1, \ldots, \tilde{l} - 1, \quad i = \tilde{l} + 1, \ldots, n-1, \\ 1 < \tilde{l} \leq n-1, \quad \nu = 0 \ldots, n-1. \end{array}$$

iii) $\lambda_0 = \lambda_{i_j} \hat{k}_{0i_j}$ *and* $\hat{k}^2_{0i_j} = r$ *with* $r, i_j \in \{1, \ldots, n-1\}$, $j = 1, \ldots, r$. *For first approximation the following Lie symmetry generators are obtained:*

$$< T_\nu,\ \hat{k}^{-1}_{0a} L_{0j} - \hat{k}^{-1}_{0j} L_{0a} + R_{ja},\ \hat{k}^{-1}_{0a} L_{0\tilde{r}} + R_{\tilde{r}a}\ (r \neq n-1),$$

$$R_{cb}(r \neq \{n-1,\, n-2\},\ S_x + \sum_{j=1}^{r} \hat{k}^{-1}_{0j} L_{0j},\ S_1,\ \Gamma_{01},\ \tilde{\Lambda}^{4/(n-2)}_{x_0 \ldots x_{n-1} 1},\ \Upsilon_{x_0 \ldots x_{n-1}} > .$$

For k-order approximations ($k \geq 2$) it follows that

$$< T_\nu,\ \hat{k}^{-1}_{0a} L_{0j} - \hat{k}^{-1}_{0j} L_{0a} + R_{ja},\ \hat{k}^{-1}_{0a} L_{0\tilde{r}} + R_{\tilde{r}a}\ (r \neq n-1),$$

$$R_{cb}(r \neq \{n-1,\, n-2\}),\ S_x + \sum_{j=1}^{r} \hat{k}^{-1}_{0j} L_{0j},\ S_k,\ \Gamma_{0k},\ \tilde{\Lambda}^{4/(n-2)}_{x_0 \ldots x_{n-1} k} > .$$

Here

$$\begin{array}{l} a = 1, \ldots, r-1, \quad j = a+1, \ldots, r, \quad \tilde{r} = r+1, \ldots, n-1, \\ b = r+1, \ldots, n-2, \quad c = b+1, \ldots, n-1. \end{array}$$

iv) $\lambda_0 = \lambda_{i_j} \hat{k}_{0i_j}$, $\hat{k}^2_{0i_j} \neq r$ *and* $\lambda_0^2 - \lambda_1 - \cdots - \lambda^2_{n-1} = 0$ *with* $r, i_j \in \{1, \ldots, n-1\}$, $j = 1, \ldots, r$. *For first approximation (22) admits the following Lie symmetry generators*

$$< T_\nu,\ L_{0j} + \hat{k}_{0j} R_{\tilde{r}j}\ (r \neq n-1),\ \left(\frac{\hat{k}_{01}}{r - \hat{k}^2_{01}} \right) S_x + \left(\frac{\hat{k}_{0i}}{\hat{k}^2_{01} - \hat{k}^2_{0i}} \right) R_{i1} + L_{01},$$

$$\left(\frac{\hat{k}^2_{0i} - \hat{k}^2_{01}}{\hat{k}_{0i}} \right) S_x - \frac{\hat{k}_{01}}{\hat{k}_{0i}} \left(\frac{r - \hat{k}^2_{0i}}{r - \hat{k}^2_{01}} \right) L_{01} + L_{0i},$$

$$R_{a(r+1)}\ (r \neq \{n-1,\, n-2\}),\ S_1,\ \Gamma_{01},\ \tilde{\tilde{\Lambda}}^{4/(n-2)}_{x_0 \ldots x_{n-1} 1},\ \Upsilon_{x_0 \ldots x_{n-1}} > .$$

For k-order approximations ($k \geq 2$) we obtain

$$< T_\nu,\ L_{0j} + \hat{k}_{0j} R_{\tilde{r}j}\ (r \neq n-1),\ \left(\frac{\hat{k}_{01}}{r - \hat{k}^2_{01}} \right) S_x + \left(\frac{\hat{k}_{0i}}{\hat{k}^2_{01} - \hat{k}^2_{0i}} \right) R_{i1} + L_{01},$$

$$\left(\frac{\hat{k}^2_{0i} - \hat{k}^2_{01}}{\hat{k}_{0i}} \right) S_x - \frac{\hat{k}_{01}}{\hat{k}_{0i}} \left(\frac{r - \hat{k}^2_{0i}}{r - \hat{k}^2_{01}} \right) L_{01} + L_{0i},$$

$$R_{a(r+1)}\ (r \neq \{n-1,\, n-2\}),\ S_k,\ \Gamma_{0k},\ \tilde{\tilde{\Lambda}}^{4/(n-2)}_{x_0 \ldots x_{n-1} k} > .$$

Here

$$j=1,\dots,r,\quad i=2,\dots,r,\quad \tilde{r}=r+1,\dots,n-1,\quad a=r+2,\dots,n-1.$$

The proof is performed in the Appendix. Let us define the symbols used in theorem 6 that were not defined before:

$$\Lambda^{4/(n-2)}_{(x_0\pm x_l)k}=\{2\lambda_0c_2(x_0\pm x_l)u_0-(n+2)u_1\}\frac{\partial}{\partial u_k}+(2-n)u_0\frac{\partial}{\partial u_{k-1}},\qquad(23)$$

$$\tilde{\Lambda}^{4/(n-2)}_{x_0\dots x_{n-1}k}=\left\{2\lambda_0c_2\left(x_0-\sum_{j=1}^{n-1}\hat{k}^{-1}_{0j}x_j\right)u_0-(n+2)u_1\right\}\frac{\partial}{\partial u_k}+$$
$$(2-n)u_0\frac{\partial}{\partial u_{k-1}},\qquad(24)$$

$$\tilde{\tilde{\Lambda}}^{4/(n-2)}_{x_0\dots x_{n-1}k}=\left\{\left(\frac{6-n}{2}\right)\lambda_0c_2\left(x_0-\sum_{j=1}^{n-1}\hat{k}^{-1}_{0j}x_j\right)u_0-(2+n)u_1\right\}\frac{\partial}{\partial u_k}+$$
$$(2-n)u_0\frac{\partial}{\partial u_k-1},\ (25)$$

$$\Upsilon_{(x_0\pm x_l)}=K_{0u_0}\pm K_{lu_0}+\left\{(2-n)(x_0\pm x_l)u_1+\lambda_0c_2(x_0\pm x_l)^2u_0\right\}\frac{\partial}{\partial u_1},\ (26)$$

$$\Upsilon_{x_0\dots x_{n-1}}=K_{0u_0}-\sum_{j=1}^{n-1}\hat{k}^{-1}_{0j}K_{ju_0}+\qquad(27)$$
$$\left\{(2-n)\left(x_0-\sum_{j=1}^{n-1}\hat{k}^{-1}_{0j}x_j\right)u_1++\lambda_0c_2\left(x_0-\sum_{j=1}^{n-1}\hat{k}^{-1}_{0j}x_j\right)^2u_0\right\}\frac{\partial}{\partial u_1}.$$

From theorem 6 it is clear that approximate conformal invariance of (1) exists only for first approximation.

REMARK: *The function f in theorem 6 can be given in the more general form*

$$f(u)=(c_1u+c_2)^{4/(n-2)}$$

if we allow the infinitesimal function η_0 of $\partial/\partial u_0$ in the generator K_{0u_0} and K_{lu_0} to depend on c_1 and c_2.

4 Conditional Symmetries of the Approximate System

In this section we study conditional symmetries of (12) in the form of Q-symmetries. Q-symmetries are obtained by constructing symmetries of the form

$$Q = \sum_{\mu=0}^{n-1} \xi_\mu(x_0, \ldots, x_{n-1}, u_0, \ldots, u_k) \frac{\partial}{\partial x_\mu} + \sum_{j=0}^{k} \eta_j(x_0, \ldots, x_k, u_0, \ldots, u_k) \frac{\partial}{\partial u_j}$$

under the condition

$$\eta_j - \sum_{\mu=0}^{n-1} \xi_\mu \frac{\partial u_j}{\partial x_\mu} = 0,$$

where $j = 0, \ldots, k$.

For first approximation of (12) we study the system

$$\begin{aligned} &\Box_n u_0 + m^2 u_0 = 0, \\ &\Box_n u_1 + m^2 u_1 - (1 - u_0^2) \frac{\partial u_0}{\partial x_0} = 0. \end{aligned}$$

Five classes of Q-symmetries follow:

Class 1: Here $m^2 = -\varkappa^2$, $(\varkappa \in \mathcal{R})$.

$$\begin{aligned} Q_{11} &= \frac{\partial}{\partial x_0} + e^{\varkappa x_0} \frac{\partial}{\partial u_0} + \left(-\frac{1}{2\varkappa^2} e^{2\varkappa x_0} u_0 + \varkappa u_1 + v_1(x_0, \ldots, x_{n-1}) \right) \frac{\partial}{\partial u_1}, \\ Q_{12} &= \frac{\partial}{\partial x_0} + \{ x_0 u_0 + v_2(x_0, \ldots, x_{n-1}) + \varkappa u_1 \} \frac{\partial}{\partial u_1}, \\ Q_{13} &= \frac{\partial}{\partial x_0} + \{ u_0 + \varkappa u_1 + v_2(x_0, \ldots, x_{n-1}) \} \frac{\partial}{\partial u_1}, \end{aligned}$$

where v_1 and v_2 are solutions of the linear equations

$$\Box_n v_1 - v_1 \varkappa^2 - \frac{2e^{3x_0\varkappa}}{\varkappa} = 0, \tag{28}$$

$$\Box_n v_2 - v_2 \varkappa^2 = 0, \tag{29}$$

respectively.

Class 2:

$$\begin{aligned} Q_{21} &= \frac{\partial}{\partial x_0} + \{ x_0 u_0 + v_3(x_0, \ldots, x_{n-1}) \} \frac{\partial}{\partial u_1}, \\ Q_{22} &= \frac{\partial}{\partial x_0} + \{ u_0 + v_3(x_0, \ldots, x_{n-1}) \} \frac{\partial}{\partial u_1}, \\ Q_{23} &= \frac{\partial}{\partial x_0} + \{ u_1 + v_3(x_0, \ldots, x_{n-1}) \} \frac{\partial}{\partial u_1}, \end{aligned}$$

where v_3 is a solution of the following equation:

$$\Box_n v_3 + m^2 v_3 = 0. \tag{30}$$

Class 3:

$$\begin{aligned}
Q_{31} &= \frac{\partial}{\partial x_0} + \varkappa u_0 \frac{\partial}{\partial u_0} + \{-\varkappa x_0 u_0 + u_0 + 3\varkappa u_1 + v_3(x_0, \ldots, x_{n-1})\} \frac{\partial}{\partial u_1}, \\
Q_{32} &= \frac{\partial}{\partial x_0} + \varkappa u_0 \frac{\partial}{\partial u_0} + \\
&\left\{ 3\varkappa u_1 - \left(\frac{1}{2\varkappa} e^{-2\varkappa x_0} + \varkappa x_0 \right) u_0 + v_3(x_0, \ldots, x_{n-1}) \right\} \frac{\partial}{\partial u_1},
\end{aligned}$$

where v_3 is a solution of (30).
Class 4: Here $\omega^2 = -4\varkappa^2$, $(\varkappa \in \mathcal{R})$.

$$\begin{aligned}
Q_{41} &= \frac{\partial}{\partial x_0} + \left\{ \varkappa u_0 + e^{-2\varkappa x_0} \right\} \frac{\partial}{\partial u_0} + \\
&\left\{ 3\varkappa u_1 - \left(\varkappa x_0 + \frac{1}{4\varkappa^2} e^{-4\varkappa x_0} \right) u_0 + v_4(x_0, \ldots, x_{n-1} \right\} \frac{\partial}{\partial u_1},
\end{aligned}$$

$$\begin{aligned}
Q_{42} &= \frac{\partial}{\partial x_0} + \varkappa u_0 \frac{\partial}{\partial u_0} + \\
&\left\{ 3\varkappa u_0 - \left(\varkappa x_0 u_0 + \frac{1}{2\varkappa} e^{-2\varkappa x_0} \right) u_0 + v_5(x_0, \ldots, x_{n-1}) \right\} \frac{\partial}{\partial u_1},
\end{aligned}$$

$$Q_{43} = \frac{\partial}{\partial x_0} + \varkappa u_0 \frac{\partial}{\partial u_0} + \{-(1 + \varkappa x_0) u_0 + 3\varkappa u_1 + v_5(x_0, \ldots, x_{n-1}\} \frac{\partial}{\partial u_1},$$

where v_4 and v_5 are solutions of the equations:

$$\Box_n v_4 - 4 v_4 \varkappa^2 + 2\varkappa e^{-2\varkappa x_0} + \frac{2}{\varkappa} e^{-6\varkappa x_0} = 0, \tag{31}$$

$$\Box_n v_5 - 4 v_5 \varkappa^2 = 0, \tag{32}$$

respectively.
Remark: *The (infinite) Q-symmetries in every class given above belong to the same Q-condition so that subalgebra classifications of the Q-symmetries that belong to the same class can be considered to construct exact solutions.*

As an example we consider Q_{11}, given in class 1, where a solution of (28) is given by

$$\begin{aligned}
v_1(x_0, \ldots, x_{n-1}) = \sum_{i=1}^{\infty} c_i \exp\left(\sum_{j=1}^{n-1} b_{ij} x_j + x_0 \sqrt{\sum_{j=1}^{n-1} b_{ij}^2 + \varkappa^2} \right) + \\
\sum_{i=1}^{\infty} c_i' \exp\left(\sum_{j=1}^{n-1} b_{ij}' - x_0 \sqrt{\sum_{j=1}^{n-1} b_{ij}'^2 + \varkappa^2} \right) + \frac{1}{4\varkappa^3} e^{3\varkappa x_0}.
\end{aligned}$$

Here c_i, c_i' and b_{ij}, b_{ij}' are arbitrary constants. By solving the associated Lagrange system we obtain the following ansatz:

$$
\begin{aligned}
u_0 &= \varphi_0(\omega) + \frac{1}{\varkappa} e^{\varkappa x_0}, &(33)\\
u_1 &= -\frac{1}{2\varkappa^3} e^{2\varkappa x_0}\varphi_0(\omega) + e^{\varkappa x_0}\varphi_1(\omega) + \\
\sum_{i=1}^{\infty} & \frac{c_i}{\sqrt{\sum_{j=1}^{n-1} b_{ij}^2 + \varkappa^2} - \varkappa} \exp\left(\sum_{j=1}^{n-1} b_{ij}x_j + x_0\sqrt{\sum_{j=1}^{n-1} b_{ij}^2 + \varkappa^2}\right) + &(34)\\
\sum_{i=1}^{\infty} & \frac{\tilde{c}_i'}{\sqrt{\sum_{j=1}^{n-1} b_{ij}'^2 + \varkappa^2} - \varkappa} \exp\left(\sum_{j=1}^{n-1} b_{ij}'x_j + x_0\sqrt{\sum_{j=1}^{n-1} b_{ij}'^2 + \varkappa^2}\right) - \frac{1}{8\varkappa^2} e^{3\varkappa x_0},
\end{aligned}
$$

where

$$
\begin{aligned}
\varphi_0(\omega) &= \frac{\tilde{c}_1}{\varkappa\sqrt{n-1}} \sin\left\{\frac{\varkappa}{\sqrt{n-1}}(\omega + \tilde{c}_2)\right\},\\
\varphi_1(\omega) &= \left(\frac{\tilde{c}_1}{4\varkappa^2} - \frac{1}{2(n-1)}\right)\omega^2 + \\
&\left(\frac{\tilde{c}_1\tilde{c}_2}{2\varkappa^2} + \tilde{c}_3\right)\omega + \frac{(n-1)\tilde{c}_1^2}{8\varkappa^4} \cos\left\{\frac{2\varkappa}{\sqrt{n-1}}(\omega + \tilde{c}_2)\right\} + \tilde{c}_4.
\end{aligned}
$$

Here $\tilde{c}_j$ are arbitrary constants and φ_0 and φ_1 are the general solutions of the reduced equations

$$
(n-1)\frac{d^2\varphi_0}{d\omega^2} + \varkappa^2\varphi_0 = 0, \qquad (n-1)\frac{d^2\varphi_1}{d\omega^2} + 1 - \varphi_0^2 = 0,
$$

where $\omega = \sum_{j=1}^{n-1} x_j$.

A first approximate solution of

$$
\Box_n u + m^2 u - \varepsilon(1 - u^2)\frac{\partial u}{\partial x_0} = 0
$$

is then given by $u = u_0 + \varepsilon u_1$, where u_0 and u_1 are given by (33) and (34), respectively. An analogous procedure can be followed for all other cases in section 3.

Appendix: Proof of Theorem 6

We consider only the first approximation with $\lambda_0 \neq 0$ and $n \neq 2$. In order to prove the necessity we consider the invariance condition in the form

$$
X^{(2)}S_0\Big|_{S_0=0,\,S_1=0} = 0, \qquad X^{(2)}S_1\Big|_{S_0=0,\,S_1=0} = 0,
$$

where

$$S_0 \equiv \Box_n u_0 = 0,$$
$$S_1 \equiv \Box_n u_1 - f(u_0)\,(\lambda_0 u_{0,0} + \lambda_{i_1} u_{0,i_1} + \cdots + \lambda_{i_r} u_{0,i_r})$$

(with $r, i_1, \ldots, i_r \in \{1, \ldots, n-1\},\ r = i_1 + \cdots + i_r$) and $X^{(2)}$ is the second prolongation of the Lie symmetry generator X, with

$$X = \sum_{\mu=0}^{n-1} \xi_\mu(x_0, \ldots, x_{n-1}, u_0, u_1)\frac{\partial}{\partial x_\mu} + \sum_{i=0}^{1} \eta_i(x_0, \ldots, x_{n-1}, u_0, u_1)\frac{\partial}{\partial u_i}.$$

We make use of the following notation:

$$\xi_{i,j_1j_2} \equiv \frac{\partial^2 \xi_i}{\partial x_{j_1}\partial x_{j_2}}, \quad \eta_{i,j_1u_{j_2}} \equiv \frac{\partial^2 \eta_i}{\partial x_{j_1}\partial u_{j_2}}, \quad \text{etc.}$$

The determining equations for the unknown functions ξ and η are the following:

$$\begin{array}{rcll}
2\eta_{0,0u_0} & - & \Box_n\xi_0 + \lambda_0 f\eta_{0,u_1} = 0, & \quad (1.1)\\
2\eta_{0,ju_0} & + & \Box_n\xi_j - \lambda_j f\eta_{0,u_1} = 0, & j = 1, \ldots, r, \quad (1.2)\\
2\eta_{0,iu_0} & + & \Box_n\xi_i = 0, & i = r+1, \ldots, n-1,\\
 & & & r < n-1, \quad (1.3)\\
\eta_{0,\mu u_1} & = & 0, & \mu = 0, \ldots, n-1, \quad (1.4)\\
\eta_{0,u_0u_0} & = & \eta_0, u_1u_1 = 0, \quad \Box_n\eta_0 = 0, & \quad (1.5)\\
\xi_{0,0} & - & \xi_{j,j} = 0, & \\
\xi_{0,j} & - & \xi_{j,0} = 0, & j = 1, \ldots, n-1, \quad (1.6)\\
\xi_{i,j} & + & \xi_{j,i} = 0, & i \neq j = 1, \ldots, n-1, \quad (1.7)\\
2\eta_{1,0u_0} & + & f\Big(\lambda_0\eta_{1,u_1} - 2\lambda_0\xi_{0,0} - \lambda_0\eta_{0,u_0} + & \\
 & & \displaystyle\sum_{i=0}^{r}\lambda_i\xi_{0,i}\Big) - \lambda_0\eta_0\frac{df}{du_0} = 0, & \quad (1.8)\\
-2\eta_{1,0u_j} & + & f\Big(\lambda_j\eta_{1,u_1} - 2\lambda_j\xi_{0,0} - \lambda_j\eta_{0,u_0} + & \\
 & & \displaystyle\sum_{i=0}^{r}\lambda_i\xi_{j,i}\Big) - \lambda_j\eta_0\frac{df}{du_0} = 0, & j = 1, \ldots, r, \quad (1.9)\\
-2\eta_{1,iu_0} & + & f\displaystyle\sum_{p=0}^{r}\lambda_p\xi_{i,p} = 0, & i = r+1, \ldots, n-1,\\
 & & & r < n-1, \quad (1.10)\\
2\eta_{1,0u_1} & - & \Box_n\xi_0 - \lambda_0 f\eta_{0,u_1} = 0, & \quad (1.11)\\
2\eta_{1,ju_1} & - & \Box_n\xi_j - \lambda_j f\eta_{0,u_1} = 0, & j = 1, \ldots, r, \quad (1.12)
\end{array}$$

$$2\eta_{1,iu_1} - \Box_n \xi_i = 0, \qquad i = r+1, \ldots, n-1, \quad r < n-1, \tag{1.13}$$

$$\eta_{1,u_1u_1} = \eta_{1,u_0u_0} = 0, \tag{1.14}$$

$$\Box_n \eta_1 - f \sum_{p=0}^{r} \lambda_p \eta_{0,p} = 0. \tag{1.15}$$

From equations (1.1) – (1.7) and (1.13) – (1.15) the following expressions for ξ_μ and η_j are obtained:

$$\xi_\mu = 2x_\mu \left(b_0 x_0 - \sum_{i=1}^{n-1} b_i x_i \right) - b_\mu \left(x_0^2 - \sum_{i=1}^{n-1} x_i^2 \right) + \sum_{\mu,\nu=0}^{n-1} c_{\mu\nu} x_\nu + d_\nu, \tag{2.1}$$

$$\eta_0 = \left\{ (2-n) \left(b_0 x_0 - \sum_{i=1}^{n-1} b_i x_i \right) + \alpha_{00} \right\} u_0, \tag{2.2}$$

$$\eta_{1,u_1} = (2-n) \left(b_0 x_0 - \sum_{i=1}^{n-1} b_i x_i \right) + \alpha_{11}, \tag{2.3}$$

where $c_{0a} = c_{a0}$, $c_{ab} = -c_{ba}$, $c_{00} = c_{aa}$, $a \neq b = 1, \ldots, n-1$, and $\mu = 0, \ldots, n-1$ ($\alpha_{\mu\nu}, b_\mu, c_{\mu\nu}, d_\mu \in \mathcal{R}$). The expressions (2.1) – (2.3), with equations (1.8) – (1.10), results in the following set of equations:

$$\lambda_0 b_0 (n-4) \frac{df}{du_0} - 2 \frac{df}{du_0} \left(\sum_{j=1}^{r} \lambda_j b_j \right) + \lambda_0 b_0 (n-2) u_0 \frac{d^2 f}{du_0^2} = 0, \tag{3.1}$$

$$\lambda_j b_j (n-4) \frac{df}{du_0} - 2 \frac{df}{du_0} \left(\lambda_0 b_0 - \sum_{s \neq j,\, s=1}^{r} \lambda_s b_s \right) + \\ + \lambda_j b_j (n-2) u_0 \frac{d^2 f}{du_0^2} = 0, \qquad j = 1, \ldots, r, \tag{3.2}$$

$$\lambda_i b_p (n-4) \frac{df}{du_0} - 2\lambda_p b_i \frac{df}{du_0} + \\ \lambda_i b_p (n-2) u_0 \frac{d^2 f}{du_0^2} = 0, \qquad i \neq p = 0, \ldots, r, \tag{3.3}$$

$$\lambda_i b_q (n-4) \frac{df}{du_0} + \lambda_i b_q (n-2) u_0 \frac{d^2 f}{du_0^2} = 0, \qquad i = 0, \ldots, r, \quad q = r+1, \ldots, n-1, \quad r < n-1, \tag{3.4}$$

$$\lambda_i \frac{df}{du_0}(\alpha_{11} - 2\alpha_{00} - c_{00}) + \frac{df}{du_0}\left(\sum_{p\neq i,\, p=0}^{r} \lambda_p c_{ip}\right) -$$
$$\lambda_i \alpha_{00} u_0 \frac{d^2 f}{du_0^2} = 0, \qquad i = 0, \ldots, r, \tag{3.5}$$

$$\frac{df}{du_0}\left(\lambda_0 b_0 - \sum_{j=1}^{r} \lambda_j b_j\right) = 0, \tag{3.6}$$

$$\frac{df}{du_0}\lambda_i b_p = 0, \qquad i = 0, \ldots, r, \quad p = r+1, \ldots, n-1, \quad r < n-1 \tag{3.7}$$

$$\frac{df}{du_0}\left(\lambda_0 c_{0p} - \sum_{j=1}^{r} \lambda_j c_{jp}\right) = 0, \qquad p = r+1, \ldots, n-1, \quad r < n-1. \tag{3.8}$$

For nonconstant functions f, (3.6) results in the following condition:

$$\lambda_0 b_0 - \sum_{j=1}^{r} \lambda_j b_j = 0.$$

For $b_0 = 0$ it follows that $b_j = 0$ for all $j = 1, \ldots, n-1$, i.e., there can exist no conformal symmetry generator. We can thus assume that $b_0 \neq 0$. Note that from (3.7) it follows that $b_p = 0$ for $p = r+1, \ldots, n-1$, where $r < n-1$. From the above condition and eqs. (3.1) – (3.2) we obtain

$$(n-6)\frac{df}{du_0} - (n-2)u_0 \frac{d^2 f}{f u_0^2} = 0,$$

i.e.,

$$f(u_0) = c_1 u_0^{4/(n-2)} + c_2.$$

From the above function f, together with (3.1)–(3.3), the following relation is obtained:

$$\frac{\lambda_0}{\lambda_j} = \frac{b_0}{b_j} \equiv \hat{k}_{0j}, \quad j = 1, \ldots, r$$

i.e.,

$$\lambda_l b_l (\hat{k}_{0l}^2 - 1) - \sum_{j\neq l,\, j=1}^{r} \lambda_j b_j = 0, \quad l = 1, \ldots, r.$$

This results in two essentially different cases:

I) $\hat{k}_{0l}^2 = 1$, $\sum_{j\neq l,\, j=1}^{r} \lambda_j b_j = 0$.

II) $\hat{k}_{0l}^2 \neq 1$, $\lambda_l b_l(\hat{k}_{0l}^2 - 1) - \sum_{j\neq l,\, j=1}^{r} \lambda_j b_j = 0$.

Let us consider case (I). This can be rewritten as

$$b_1^2 + \cdots + b_{l-1}^2 + b_{l+1}^2 + \cdots + b_r = 0,$$

i.e., $b_j = \lambda_j = 0,\ j = 1, \ldots, \widehat{l}, \ldots, r,\ 0 < l \leq r$. Thus it follows that

$$\lambda_0 = \pm\lambda_l, \quad 0 < l \leq r.$$

This is written as case (i) and (ii) in theorem 6. From case (II) we can obtain case (iii) and (iv) of theorem 6. By inserting the above relations and function f in the determining equations we obtain the generators listed in theorem 6. This proofs the necessary condition. By substituting the obtained function f (with the related conditions) into the wave equation the sufficient condition is satisfied. For k-order approximations one has to consider mathematical induction.

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