

# Homogeneous Manifold, Loop Algebra, Coupled KdV System and Generalised Miura Transformation

*I. MUKHOPADHYA and A. Roy CHOWDHURY*

*High Energy Physics Division,  
Department of Physics, Jadavpur University  
Calcutta 700 032, India*

*Submitted by P. OLVER  
Received March 20, 1994*

## Abstract

Coupled *KdV* equations are deduced by considering the homogeneous manifold corresponding to the homogeneous Heisenberg subalgebra of the Loop group ( $L(S^1, SL(2, C))$ ). Utilisation of Birkhoff decomposition and further subalgebra consideration leads to a new generalised form of Miura map and two sets of modified equations. A second set of Miura transformation can also be generated leading to complicated form of coupled integrable systems.

## 1 Introduction

The use of Lie algebra and Lie groups in the study of integrable systems have proved to be very fruitful over the last two decade. While the *AKNS* [1] approach leads to a variety of equations all at a time, the variant of the method [2] using homogeneous spaces of Lie algebra and its affine version leads to a specific form only. In this respect the basic formulations those exist are due to Adler [3], Kostant [4], Symms [5], Guil [6], and others. While some uses the co-adjoint action on the affine Lie algebra as the starting point, others go for the zero curvature equation in conjunction with the classical *r*-matrix [8]. Each of the methods have the respective advantage and disadvantages. A closely related approach is that of pseudo-differential operators [9] whose relation with the matrix Lax pair has been exhaustively studied in the famous article of Drinfeld and Sokolov [10]. It may be worth mentioning that the Miura-map has played a prominent role in these formulations and in the study of nonlinear *KdV* system as a whole. It is the aim of the present work to

show how to construct coupled  $KdV$  systems by considering the homogeneous manifold of Heisenberg subalgebra (homogeneous) and to derive a set of generalised. Miura like transformations from the factorization problem. This is achieved here through a chain of decomposition of the Lie algebra  $SL(2, C)$ -valued holomorphic functions on the unit disc. The paper is organized as follows. In section two we have shown how the considerations of the homogeneous Heisenberg subalgebra leads to the coupled  $KdV$  system, through the use of the zero curvature representation. In section three the factorization property of the Lax eigenfunction is invoked and a generalised Miura map is deduced. Actually we have two such maps each leading to a different set of modified coupled set of equations. In the last section we show how a further consideration of the factorization problem associated with the obtained modified equation leads to second modified class of equations which are also integrable.

## 2 Coupled $KdV$ System and Loop Groups

We denote by  $g = L(S^1, SL(2, C))$  the Lie algebra of smooth functions on the circle  $S^1 = \{\lambda \in \mathcal{C} : |\lambda| = 1\}$  with values in the complex Lie algebra of  $2 \times 2$  matrices of zero trace. Such an algebra is the loop algebra. The boundary values on the circle  $S^1$  of the holomorphic function over the Riemann sphere  $|\lambda| < 1$  and  $|\lambda| > 1$  respectively allow us to define the subalgebras  $g_+$  and  $g_-$ , with the direct sum decomposition of  $g$  as

$$g = g_+ \oplus g_-.$$

In terms of fourier series expansion of an element  $X$  we have

$$X(\lambda) = \sum x - n\lambda^n, \quad X \in SL(2, c) \quad \text{and} \quad X(\lambda) = X_+ - X_-$$

the above decomposition then reads:

$$X_+(\lambda) = \sum_{n \geq 0} X_n \lambda^n, \quad X_-(\lambda) = - \sum_{n < 0} X_n \lambda^n$$

the loop group  $G = L(S^1, SL(2, C))$  of smooth functions of  $S^1$  with values in  $2 \times 2$  complex matrices (having determinant equal to unity) admits  $g$  as its Lie-algebra. The subgroups pertaining to  $g_+$ ,  $g_-$  are defined similarly.

The element  $\psi$  of  $G$  for which there exists  $\psi_+ \in G_+$  and  $\psi_- \in G_-$ , obeys the factorization condition

$$\psi = \psi_-^{-1} \cdot \psi_+ \tag{1}$$

in the sense of Birkhoff [11]. Using the factorizability assumption we can map the point  $\psi$ ,  $G_+$  into the element  $\psi_- \in G_-$ . A consequence of this is that we have a co-ordinate system, the Lie algebra  $g$ , for the set of points  $\psi \cdot G_+$  in  $G/G_+$  associated to factorizable elements  $\psi$  in  $G$ .

Let us assume that  $p$  be the  $G$  valued function of the two complex variable  $x$  and  $t$  given by

$$p(x, t) = \exp(xH + tH^3) \tag{2}$$

where  $H \in g$  the element  $\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$ .

The centralizer of  $H$  in  $g$  is called the homogeneous Heisenberg subalgebra and consists of the series of the form  $t_n H^n$ , for odd,  $t_n \in \mathcal{O}$ . Remark that for  $n$  even  $H^n \sim 1$  set

$$\psi = p \cdot g \tag{3}$$

where  $g$  is a constant element independent of  $x$  and  $t$ . From

$$\psi = \psi_-^{-1} \cdot \psi_+ \tag{4}$$

On taking right differential we get

$$d\psi_- \cdot \psi_-^{-1} + Ad\psi_- \cdot \Gamma = d\psi_+ \psi_-^{-1} \tag{5}$$

with

$$\Gamma = d\psi \cdot \psi^{-1} = Hdx + H^3 dt. \tag{6}$$

From equation (5)

$$\begin{aligned} \partial_X \psi_- \cdot \psi_-^{-1} + P_-(Ad\psi_- \cdot H) &= 0, \\ \partial_t \psi_- \cdot \psi_-^{-1} + P_-(Ad\psi_- \cdot H^3) &= 0 \end{aligned} \tag{7}$$

where  $P_-$  denotes projection on  $g_-$  parallel to  $g_+$ . We can also write

$$\psi_- = U' Z' \tag{8}$$

where

$$U' = \exp(U), \quad Z' = \exp(Z)$$

with

$$U(\lambda) = \begin{bmatrix} 0 & \lambda^{-1}(r + B) \\ -C & 0 \end{bmatrix}; \quad Z(\lambda) = \begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix}.$$

So that  $U$  with  $r = 0 \in g_- \cap \text{Im } adH$  and  $Z$  takes values in  $g_- \cap \text{Ker } adH$  as can be checked by easy computation. Now, from equation (7)

$$\partial_x U' \cdot U'^{-1} + AdU' \cdot (\partial_x Z) + P_-(AdU' \cdot H) = 0 \tag{9}$$

which reduces to

$$\begin{aligned} \partial_x U + \frac{1}{2}[U \cdot \partial_x U] + \frac{1}{3!}[U, [U, \partial_x U]] + \dots + \partial_x Z + [U, \partial_x Z] + \\ \frac{1}{2}[U, [U, \partial_x Z]] + \dots + P_- \left( H + [U, H] + \frac{1}{2}[U, [U, H]] + \dots \right) = 0. \end{aligned} \tag{10}$$

We now set

$$A = \sum A_n \lambda^{-n}, \quad B = \sum B_n \lambda^{-n}, \quad C = \sum \lambda^{-n} C_n$$

whence equation (10) leads to

$$r_t - \frac{1}{4}r_{xxx} + 6r r_x C_1 = 0 \quad (11)$$

and

$$C_{1t} - \frac{1}{4}C_{1xxx} + 6rC_1C_{1x} = 0$$

which is the required coupled  $KdV$  system.

Now to check the consistency of the equation set (11) we revert to the other half of the factorization equation. We calculate,

$$\omega_+ = P_+(Ad\psi_- \cdot \Gamma) \quad (12)$$

this yields:

$$\omega_+ = Mdx + Ndt \quad (13)$$

with

$$M = P_+(Ad\psi_- \cdot H) = P_+ [e^U \cdot H e^{-U}] = \begin{pmatrix} \lambda & -2r \\ -2C_1 & -\lambda \end{pmatrix} \quad (14)$$

along with

$$N = P_+(Ad\psi_- \cdot H^3) = P_+ [\lambda^2(e^U H e^{-U})] = \begin{bmatrix} \lambda^3 - 2[(\lambda C_1 + C_2)r] + B_1 C_1 & -2(\lambda^2 r + \lambda B_1 + B_2) + 4/3r^2 C_1 \\ -2(\lambda^2 C_1 + \lambda C_2 + C_3) + 4/3r C_1^2 & -\lambda^3 + 2[r(\lambda C_1 + C_2) + B_1 C_1] \end{bmatrix}. \quad (15)$$

So that the zero curvature condition

$$N_t - N_x + [M, N] = 0 \quad (16)$$

immediately implies,

(i) that the equations coming from the diagonal elements of (16) are identically satisfied.

(ii) the (12) and (21) elements lead to the equations (11).

### 3 Generalised Miura Transformation

From equation (14) and (15) we can at once

$$\partial_x \psi_- \cdot \psi_-^{-1} + P_-(Ad\psi_- \cdot M) = 0, \quad (17)$$

$$\partial_t \psi_- \cdot \psi_-^{-1} + P_-(Ad\psi_- \cdot N) = 0.$$

Now we invoke the decomposition

$$g = g_+ \oplus g_- \quad (18)$$

for a fixed  $\lambda_1 \in D(0, 1)$ , where  $D$  represents the unit disc with centre at the origin, so that

$$g_- = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}, \quad c \in \mathcal{Q} \right\}, \tag{19}$$

$$g_+ = X \in g; \quad X(\lambda_1) = \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}, \quad a, b \in \mathcal{Q}.$$

Then we have  $\psi_- = \exp(t_- g_-) = \begin{pmatrix} 1 & 0 \\ V & 1 \end{pmatrix}$  and

$$\partial_x \psi_- \cdot \psi_-^{-1} = \begin{pmatrix} 0 & 0 \\ v_x & 0 \end{pmatrix}$$

along with

$$Ad\psi_- \cdot M = \begin{bmatrix} \lambda_1 + 2rv & -2r \\ 2v\lambda_1 - 2C_1 + 2v^2r & -(2vr + \lambda_1) \end{bmatrix} \tag{20}$$

whence the first equation of (17) gives,

$$2C_1 = v_X + 2v\lambda_1 + 2v^2r. \tag{21}$$

So that  $v$  represents the new nonlinear variable. To obtain the time flow for this new variable we concentrate on equation (17) again, this time on its time part. Which immediately leads to

$$v_t + 22vN_{11} + N_{21} - v^2N_{12} = 0 \tag{22}$$

which is seen to reduce to,

$$v_t - \frac{1}{4}v_{xxx} - 3vv_xr_x + 6v^2r^2v_x + 6rvv_x\lambda_1 = 0 \tag{23}$$

after we have used

$$N_{11} = \lambda^3 - 2\lambda rC_1 + rC_{1x} - r_xC_1 = -N_{22},$$

$$N_{12} = -2\lambda^2r - \lambda r_x - 1/2r_{xx} + 4r^2C_1,$$

$$N_{21} = -2\lambda^2C_1 + \lambda C_{1x} - 1/2C_{1xx} + 4rC_1^2.$$

The other equation of the coupled set is

$$r_t - \frac{1}{4}r_{xxx} + 3rr_x(v_x + 2v\lambda_1 + 2v^2r) = 0. \tag{24}$$

So we have obtained the Miura transformation given by equation (17) which is generalised in the sense that it contains terms cubic in the nonlinear variables. So equation (21) gives the map  $(C_1, r) \rightarrow (v, r)$ , and equations (23) and (24) represent the modified set.

We now observe that in the above Miura transformation only variable  $C_1$  is changed and  $r$  remains the same. On the otherhand if we consider another decomposition of  $g$ .

$$g = g_+ \oplus g_- \tag{25}$$

where  $g_+ \begin{pmatrix} a & 0 \\ c & -a \end{pmatrix}$  and  $g_- \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$  again for some fixed  $\lambda - 1$ . Then we get

$$\psi_- = \exp(t_- g_-) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \text{ and proceeding as} \tag{26}$$

above we get

$$2r = u_x - 2\lambda_1 u + 2u^2 C_1. \tag{27}$$

A straightforward calculation using the time part of equation (17) leads to:

$$u_t - \frac{1}{4}u_{xxx} + 6u_x^2 C_1^2 - 3uu_x C_{1x} - 6\lambda_1 uu_x C = 0 \tag{28}$$

the other equation set can be seen to be:

$$C_{1t} - \frac{1}{4}C_{1xxx} + 3C_1 C_{1x}(u_x - 2\lambda_1 u + 2u^2 C_1) = 0 \tag{29}$$

so this time we have the second Miura map (27), giving the transformation  $(C_1, r) \rightarrow (C_1, u)$  and the modified equations are given by (28), and (29).

### 4 Further Miura Map

We now consider  $\omega_+ = d\psi \cdot \psi^{-1}$  so that  $\partial_x \psi = M\psi$ ,  $M$  being given by equation (14) and  $\psi$  stands for a general eigen solution,

$$\psi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}. \tag{30}$$

Due to the condition  $\psi_x = M\psi$  this can also be written as

$$\phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \frac{1}{2r}(\lambda\phi_{11} - \phi_{11x}) & \frac{1}{2r}(\lambda\phi_{12} - \phi_{12x}) \end{pmatrix}. \tag{31}$$

Now if we take  $\psi_- = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}$ , then from the Birkhoff decomposition

$$\psi_+ = \begin{pmatrix} \phi_{11} & \phi_{12} \\ v\phi_{11} + \frac{1}{2r}(\lambda\phi_{11} - \phi_{11x}) & v\phi_{12} + \frac{1}{2r}(\lambda\phi_{12} - \phi_{12x}) \end{pmatrix}. \tag{32}$$

For  $\psi_+$  to be upper traingular we get the the following condition,

$$v = -\frac{1}{2r} \left( \lambda - \frac{\phi_{11x}^1}{\phi_{11}} \right) \tag{33}$$

where  $\phi_{11}^1 = \phi_{11}(\lambda_1)$ , where  $\lambda_1 \in D(0, 1)$ . We also impose the condition that  $\det \psi = 1$  which results in the

$$\phi_{11}\phi_{12x} - \phi_{12}\phi_{11x} = -2r. \tag{34}$$

We next evaluate  $d\psi_+ \cdot \psi_+^{-1}$ . From the contraction with the vector field  $\partial_x$  we get

$$\begin{aligned} \omega_+(\partial_x)_{11} &= 2vr + \lambda, \\ \omega_+(\partial_x)_{22} &= -(2vr + \lambda), \\ \omega_+(\partial_x)_{12} &= -2vr, \\ \omega_+(\partial_x)_{21} &= 2v(\lambda - \lambda_1) + \frac{\lambda}{2} \left( r^2 r_x - \frac{r_x}{r^2} \right), \end{aligned} \tag{35}$$

which is nothing but the space part of the one form  $\omega_+ = Qdx + Rdt$  – we also keep in mind that

$$\psi = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}. \tag{36}$$

We next take a  $\lambda_2 \in D(0, 1)$ ,  $\lambda_2 \neq \lambda_1$  and seek  $\psi_1$  once more in the form:

$$\psi_- = \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}. \tag{37}$$

This factorization equation leads to (as before)

$$\begin{aligned} v &= \left[ \frac{1}{2} \lambda_2 \left( \frac{r_x}{r^2} - r^2 r_x \right) - w_x - 2w^2 r - 2w\lambda_2 \right] \frac{1}{a}, \\ a &= 4wr + 2(\lambda_2 - \lambda_1). \end{aligned} \tag{38}$$

We now try to reconstruct the time part which implies.

$$\begin{aligned} \phi_{11t} &= N_{11}\phi_{11} + N_{12}\phi_{12}, \\ \phi_{12t} &= N_{11}\phi_{12} + N_{12}\phi_{22}, \\ \phi_{21t} &= N_{21}\phi_{11} + N_{22}\phi_{21}, \\ \phi_{22t} &= N_{21}\phi_{12} + N_{22}\phi_{22} \end{aligned} \tag{39}$$

where  $\phi_{21} = \frac{1}{2r}(\lambda\phi_{11} - \phi_{11x})$ ,  $\phi_{22} = \frac{1}{2r}(\lambda\phi_{12} - \phi_{12x})$ .

The contraction with vector field  $\partial_t$  will yield the time part, but the calculation is lengthy and tedious. We therefore quote some of the results only:

$$\begin{aligned} \omega_+(\partial_t)_{11} &= 2\lambda^2 vr + \lambda v r_x + \frac{1}{2}(v r_{xx} + r C_{1x} - r_x C_1 + \lambda^3) - \\ &\quad (\lambda r + 2r^2 v)(v_x + 2v\lambda_1 + 2v^2 r), \\ \omega_+(\partial_t)_{22} &= -\omega_+(\partial_t)_{11}, \\ \omega_+(\partial_t)_{12} &= 2r^2(v_x + 2v\lambda_1 + 2v^2 r) - 2\lambda^2 r - \lambda v_x - \frac{1}{2} r_{xx} \end{aligned} \tag{40}$$

and so on. The next step is to eliminate  $C_1$  in terms of  $r$  from these equations making these expressions to depend only on  $r$  and  $v$  only. One can then utilise the transformation given in equation (38) to eliminate  $v$  in terms of  $\omega$ . So finally everything will depend on  $(r, \omega)$  only. Use of these relations will result in a integrable equation. The said computation is very cumbersome and we have indicated the basic steps. The important point is that it is possible to carry out two successive Miura map to generate new coupled integrable systems.

## 5 Discussions

In our above analysis we have indicated how the homogeneous manifold considerations can be used to generate coupled KdV system. The factorization in the sense of Birkhoff can be used to construct different Miura maps. Finally we make the remark that in the second stage of the transformations we could have taken

$$\psi_- = \begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix} \quad (41)$$

that is  $\psi_-$  to be upper triangular whence

$$\psi = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \quad (42)$$

then pursue the process of elimination again to produce a new form of second stage Miura map as in the case of first stage Miura transformation.

One of the authors (I.M) wishes to thank C.S.I.R. (Govt. of India) for a fellowship which made this work possible.

## References

- [1] Ablowitz M.J., Kaup D.J., Newell A.C. and Segur H., *Stud. Appl. Math.*, 1974, V.53, 249.
- [2] Integrable and Super integrable Systems, Ed: B.A. Kupershmidt, World Scientific Singapore, 1990.
- [3] Adler M.J., *Invent. Math.*, 1979, V.50, 219.
- [4] Konstant B., *Adv. Math.*, 1979, V.34, 195.
- [5] Symes W.W., *Invent Math.*, 1980, V.59, 13.
- [6] Guil F., *Inv. Problems.*, 1989, V.8, 559.
- [7] Chowdhury A.Roy. and Guha P., *J. Phys.*, 1990, V.A23, L 639.
- [8] Semenov-Tianshansky M.A., *Funct. Anal. Appl.*, 1983, V.17, 259.
- [9] Gelfand I.M. and Dikei L.A., *Funct. Anal. Appl.*, 1985, V.11, 93.
- [10] Drinfeld V.G. and Sokolov V.V., *Sov. J. Maths*, 1985, V.30, 1975.
- [11] Pressely A. and Segal G., *Loop groups*, Oxford. U.P., 1986.