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# Symmetry Classification for a Coupled Nonlinear Schrödinger Equations

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**Abstract**

We do a Lie symmetry classification for a system of two nonlinear coupled Schrödinger equations. Our system under consideration is a generalization of the equations which follow from the analysis of optical fibres. Reductions of some special equations are given.

We investigate the Lie symmetry properties of the following nonlinear Schrödinger system:

$$\begin{aligned} S_1 &\equiv i\frac{\partial u}{\partial z} + i\delta\frac{\partial u}{\partial \tau} + \frac{1}{2}\frac{\partial^2 u}{\partial \tau^2} - F_1(u, v, u^*, v^*, z) = 0, \\ S_2 &\equiv i\frac{\partial v}{\partial z} - i\delta\frac{\partial v}{\partial \tau} + \frac{1}{2}\frac{\partial^2 v}{\partial \tau^2} - F_2(u, v, u^*, v^*, z) = 0, \end{aligned} \qquad (1)$$

where $^*$ indicates complex conjugation.

Details on the Lie symmetry analysis can be found in one of the following books [1 – 4]. For particular functions $F_1$ and $F_2$ (see cases 3–5) the system (1) plays an important role in the analysis of optical fibres [5–11]. Before discussing important special cases, we do a general Lie symmetry classification of (1). We do not restrict ourselves to particular symmetries nor do we assume particular functional forms of (1) in our classification. This results in two propositions which provide the conditions for the most general Lie symmeties and the associated particular forms of (1).

The invariance of (1) with respect to the Lie symmetry transformation group generated by the Lie symmetry generator

$$X = \xi_0(z,\tau)\frac{\partial}{\partial\tau} + \xi_1(z,\tau)\frac{\partial}{\partial z} + \eta_1(z,\tau,u,u^*,v,v^*)\frac{\partial}{\partial u} + \eta_2(z,\tau,u,u^*,v,v^*)\frac{\partial}{\partial u^*} + \eta_3(z,\tau,u,u^*,v,v^*)\frac{\partial}{\partial v} + \eta_4(z,\tau,u,u^*,v,v^*)\frac{\partial}{\partial v^*} \qquad (2)$$

is considered. Here the infinitesimal functions $\xi_i$ and $\eta_j$ are arbitrary complex functions, which are determined by invariance conditions

$$X^{(2)}S_1\Big|_{S_1=0,S_1^*=0,S_2=0,S_2^*=0} = 0, \quad X^{(2)}S_1^*\Big|_{S_1=0,S_1^*=0,S_2=0,S_2^*=0} = 0, \qquad (3)$$

$$X^{(2)}S_2\Big|_{S_1=0,S_1^*=0,S_2=0,S_2^*=0} = 0, \quad X^{(2)}S_2^*\Big|_{S_1=0,S_1^*=0,S_2=0,S_2^*=0} = 0, \qquad (4)$$

where $X^{(2)}$ denotes the second prolongation of the generator $X$.

Let us begin the classification with the case where $F_1 = F_2 = 0$. We can state the following

**Theorem 1** *The maximal finite Lie symmetry of the uncoupled system*

$$\begin{aligned} i\frac{\partial u}{\partial z} + i\delta\frac{\partial u}{\partial \tau} + \frac{1}{2}\frac{\partial^2 u}{\partial \tau^2} &= 0, \\ i\frac{\partial v}{\partial z} - i\delta\frac{\partial v}{\partial \tau} + \frac{1}{2}\frac{\partial^2 v}{\partial \tau^2} &= 0 \end{aligned} \qquad (5)$$

*is given by the following nine generators of*

*translations*

$$G_1 = \frac{\partial}{\partial z}, \qquad G_2 = \frac{\partial}{\partial \tau},$$

*δ-deformed field rotations*

$$G_3 = \exp(-2i\delta\tau)iv\frac{\partial}{\partial u} - \exp(2i\delta\tau)iv^*\frac{\partial}{\partial u^*},$$
$$G_4 = \exp(2i\delta\tau)iu\frac{\partial}{\partial v} - \exp(-2i\delta\tau)iu^*\frac{\partial}{\partial v^*},$$

$$G_3' = \exp(-2i\delta\tau)v\frac{\partial}{\partial u} + \exp(2i\delta\tau)v^*\frac{\partial}{\partial u^*},$$
$$G_4' = \exp(2i\delta\tau)u\frac{\partial}{\partial v} + \exp(-2i\delta\tau)u^*\frac{\partial}{\partial v^*},$$

*field-dilatations*

$$G_5 = iu\frac{\partial}{\partial u} - iu^*\frac{\partial}{\partial u^*}, \quad G_6 = iv\frac{\partial}{\partial v} - iv^*\frac{\partial}{\partial v^*},$$

$$G_5' = u\frac{\partial}{\partial u} + u^*\frac{\partial}{\partial u^*}, \quad G_6' = v\frac{\partial}{\partial v} + v^*\frac{\partial}{\partial v^*},$$

*$\delta$-deformed $z\tau$-dilatations*

$$G_7 = 2z\frac{\partial}{\partial z} + \tau\frac{\partial}{\partial \tau} - i\delta\,(\tau - \delta z)\,u\frac{\partial}{\partial u} + i\delta\,(\tau - \delta z)\,u^*\frac{\partial}{\partial u^*} + \\ i\delta\,(\tau + \delta z)\,v\frac{\partial}{\partial v} - i\delta\,(\tau + \delta z)\,v^*\frac{\partial}{\partial v^*},$$

*$\delta$-deformed Galilean boost*

$$G_8 = z\frac{\partial}{\partial \tau} + iu(\tau - \delta z)\frac{\partial}{\partial u} - iu^*(\tau - \delta z)\frac{\partial}{\partial u^*} + iv(\tau + \delta z)\frac{\partial}{\partial v} - iv^*(\tau + \delta z)\frac{\partial}{\partial v^*},$$

*$\delta$-deformed projection*

$$G_9 = 2z\tau\frac{\partial}{\partial \tau} + 2z^2\frac{\partial}{\partial z} + \\ iu(iz + \tau^2 - 2z\delta\tau + \delta^2 z^2)\frac{\partial}{\partial u} - iu^*(-iz + \tau^2 - 2z\delta\tau + \delta^2 z^2)\frac{\partial}{\partial u^*} + \\ iv(iz + \tau^2 + 2\tau\delta z + \delta^2 z^2)\frac{\partial}{\partial v} - iv^*(-iz + \tau^2 + 2z\delta\tau + \delta^2 z^2)\frac{\partial}{\partial v^*}.$$

*The non-vanishing Lie brackets are:*

$$\begin{aligned}
&[G_1, G_7] = 2G_1 + \delta^2(G_5 + G_6), \quad [G_1, G_8] = [G_2, G_7] = G_2 - \delta(G_5 - G_6),\\
&[G_1, G_9] = 2G_7 - G'_5 - G'_6, \quad [G_2, G_3] = 2\delta G'_3, \quad [G_2, G_4] = -2\delta G'_4,\\
&[G_2, G_8] = G_5 + G_6, \quad [G_2, G_9] = 2G_8, \quad [G_2, G'_3] = -2\delta G_3,\\
&[G_2, G'_4] = 2\delta G_4, \quad [G_3, G_4] = G'_5 - G'_6, \quad [G_3, G_5] = -G'_3,\\
&[G_3, G_6] = [G'_3, G'_5] = -[G'_3, G'_6] = G'_3,\\
&[G_3, G'_4] = -[G_4, G'_3] = G_6 - G_5, \quad [G_3, G'_6] = -G_3,\\
&[G_3, G'_5] = [G_6, G'_3] = -[G_5, G'_3] = G_3,\\
&[G_4, G_5] = [G'_4, G'6] = -[G'_4, G'_5] = -[G_4, G_6] = G'_4,\\
&[G_4, G'_5] = -[G_4, G'_6] = -[G_5, G'_4] = [G_6, G'_4] = -G_4,\\
&[G_7, G_8] = G_8, \quad [G_7, G_9] = 2G_9, \quad [G'_3, G'_4] = G'_6 - G'_5.
\end{aligned}$$

The proof of Theorem 1, by use of the invariance conditions (3) and (4), is a standard procedure and will not be given here. It is important to note the contribution of the real constant $\delta$ to a structure of the symmetry generator. This constant $\delta$ plays an important role in study of birefringent optical fibres [8, 11] so that in this article, we always consider $\delta \neq 0$. However, this is not a restriction and the case $\delta = 0$ is also included in our results.

Let us now study the Lie symmetry properties of (1). Using the invariance conditions (3) and (4) we obtain following restrictions on the infinitesimal functions:

$$\xi_0 = \frac{1}{2}f'_1(z)\tau + f_0(z), \tag{6}$$

$$\xi_1 = f_1(z), \tag{7}$$

$$\begin{aligned}\eta_1 &= \left\{\frac{i}{4}f_1''(z)\tau^2 + \left(if_0'(z) - \frac{i}{2}\delta f_1'(z)\right)\tau + g_1(z)\right\}u + \\ &\quad a_1(z)v\exp(-2i\delta\tau) + b_1(z,\tau), \end{aligned} \tag{8}$$

$$\begin{aligned}\eta_2 &= \left\{-\frac{i}{4}f_1''(z)\tau^2 + \left(-if_0'(z) + \frac{i}{2}\delta f_1'(z)\right)\tau + g_2(z)\right\}u^* + \\ &\quad a_2(z)v^*\exp(2i\delta\tau) + b_2(z,\tau), \end{aligned} \tag{9}$$

$$\begin{aligned}\eta_3 &= \left\{\frac{i}{4}f_1''(z)\tau^2 + \left(if_0'(z) + \frac{i}{2}\delta f_1'(z)\right)\tau + g_3(z)\right\}v + \\ &\quad a_3(z)u\exp(2i\delta\tau) + b_3(z,\tau), \end{aligned} \tag{10}$$

$$\begin{aligned}\eta_4 &= \left\{-\frac{i}{4}f_1''(z)\tau^2 + \left(-if_0'(z) - \frac{i}{2}\delta f_1'(z)\right)\tau + g_4(z)\right\}v^* + \\ &\quad a_4(z)u^*\exp(2i\delta\tau) + b_4(z,\tau). \end{aligned} \tag{11}$$

Here $f_j = f_j(z)$ $(j = 0, 1)$ are arbitrary differentiable real valued functions, whereas $a_k = a_k(z)$, $b_k = b_k(z)$, and $g_k = g_k(z)$ $(k = 1, \ldots, 4)$ are arbitrary differentiable complex-valued functions. The prime indicates ordinary differentiation with respect to $z$. These real and complex functions, as well as $F_1$ and $F_2$, are conditioned by the five systems of partial differential equations (A-1)–(A-5), given in Appendix A. On solving the latter one has to distinguish between two essentially different cases, namely $f_1''(z) \neq 0$ and $f_1''(z) = 0$. We take $b_j = 0;\ j = 1, \ldots, 4$.

**Case 1** Let $f_1''(z) \neq 0$. Before finding solutions of the system (A-1)–(A-5) we intend to prove

**Lemma 1** *If*

$$f_1''(z) \neq 0,$$

*both $f_0(z)$ and $f_1(z)$ have the following general form:*

$$\alpha_1 z^2 + \alpha_2 z + \alpha_3, \tag{12}$$

*and*
*i) for $\triangle \equiv (\alpha_2^2 - 4\alpha_1\alpha_3)/(4\alpha_1^2) = 0$*

$$f_0(z) = \lambda\delta\left(z + \frac{\alpha_2}{2\alpha_1}\right)\left\{\ln(\alpha_1 z^2 + \alpha_2 z + \alpha_3) - 2\right\} + \alpha_0 z + \beta_0, \tag{13}$$

*ii) for $\triangle > 0$*

$$\begin{aligned}f_0(z) = \lambda\delta\left(z + \frac{\alpha_2}{2\alpha_1}\right)\left\{\ln(\alpha_1 z^2 + \alpha_2 z + \alpha_3) - 2\right\} - \\ \lambda\delta\sqrt{\triangle}\ln\frac{z - \sqrt{\triangle} + \alpha_2/(2\alpha_1)}{z + \sqrt{\triangle} + \alpha_2/(2\alpha_1)} + \alpha_0 z + \beta_0,\end{aligned} \tag{14}$$

*iii) for* $\triangle < 0$

$$f_0(z) = \lambda\delta\left(z + \frac{\alpha_2}{2\alpha_1}\right)\left\{\ln(\alpha_1 z^2 + \alpha_2 z + \alpha_3) - 2\right\} +$$
$$2\lambda\delta\sqrt{|\triangle|}\arctan\frac{z + \alpha_2/(2\alpha_1)}{\sqrt{|\triangle|}} + \alpha_0 z + \beta_0. \tag{15}$$

*Here* $\alpha_k,\ \beta_0,\ \lambda \in \mathcal{R};\ k = 0, 1, 2, 3.$

**P r o o f** From conditions (A-1)–(A-5) (Appendix A) it follows that

$$f_1(z)f_1'''(z) - i\lambda_0 f_1''(z) = 0, \tag{16}$$

$$f_1(z)f_1'''(z)\left(f_0'(z) - \frac{\delta}{2}f_1'(z)\right) - f_1(z)f_0''(z)f_1''(z) + \delta\lambda f_1'(z)f_1''(z) = 0, \tag{17}$$

where $\lambda_0 \in \mathcal{C}$ and $\lambda \in \mathcal{R}$. Let $\lambda_0 \neq 0$. Now (16) can be integrated twice that leads to

$$\ln[f_1'(z) + 2i\lambda_0 f_1'(z) + \gamma_0] - \frac{2i\lambda_0}{(\gamma_0 + \lambda_0^2)^{1/2}}\arctan\frac{f_1'(z) + i\lambda_0}{(\gamma_0 + \lambda_0^2)^{1/2}} = \ln(\gamma_1 f_1).$$

Here $\gamma_0$ and $\gamma_1$ are integration constants. In order to obtain an explicit form for $f_1$ we need to consider $\lambda_0 = 0$, which results in the solution $f_1(z) = \alpha_1 z^2 + \alpha_2 z + \alpha_3$, and

$$f_0''(z) = \lambda\delta\frac{2\alpha_1 z + \alpha_2}{\alpha_1 z^2 + \alpha_2 z + \alpha_3}.$$

By integrating the above equation we obtain the three different functional forms of $f_0$ stated in the lemma. □

In order to relate the functional form of $F_1$ and $F_2$ from (1) to the Lie symmetry generators of the system (1) we make the following

**Proposition 1** *Eqs. (1) possess the Lie symmetry generator (2) with infinitesimals (6)–(11) and* $f_1$, $f_0$ *given by (12) and (13)–(15), if and only if* $F_1$ *and* $F_2$ *take on the form*

$$F_1 = u\left\{\Psi_1(\Omega_1, \Omega_2, z) - i\frac{\lambda}{f_1(z)}\ln(uv^{-1})\right\}, \tag{18}$$
$$F_2 = v\left\{\Psi_2(\Omega_1, \Omega_2, z) - i\frac{\lambda}{f_1(z)}\ln(u^{*-1}v^{-1})\right\}. \tag{19}$$

*Here* $\Omega_1 \equiv (vv^*)^{-1}$, $\Omega_2 \equiv (uu^*)^{-1}$, $\lambda$ *is an arbitrary real constant, and the complex-valued functions* $\Psi_1$, $\Psi_2$ *must satisfy conditions (A-1), (A-2), and (A-6), given in Appendix A.*

For solving the system (A-1), (A-2), and (A-6) we have to distinguish between different values of $g_j;\ j = 1, \ldots, 4$. As an example we restrict ourselves to the case

$$g_1(z) + g_2(z) \neq 0 \quad \text{and} \quad g_3(z) + g_4(z) \neq 0,$$

where $g_1(z) = g_2^*(z)$ and $g_3(z) = g_4^*(z)$. With the above assumption we consider two cases:

**Case 1.1** Let $a_j = 0$; $j = 1, \ldots, 4$. Now (A-1) and (A-2) are satisfied at once. Then we could obtain the general solution of (A-6). It together with Lemma 1 and Proposition 1 gives the following functional form of $F_1$ and $F_2$:

$$\begin{aligned}
F_1 &= \frac{u}{f_1(z)}\left\{\tilde{\Psi}_1\left(\tilde{\Omega}_1, \tilde{\Omega}_2\right) + i\lambda \int \frac{g_1(z) - g_3(z)}{f_1(z)} dz - i\lambda \ln(uv^{-1}) + ig_1(z) - \right. \\
&\quad \left. \delta(f_0(z) - \beta_0) + i\frac{\alpha_1}{2} z + \frac{\delta^2}{2} f_1(z)\right\}, \\
F_2 &= \frac{v}{f_1(z)}\left\{\tilde{\Psi}_2\left(\tilde{\Omega}_1, \tilde{\Omega}_2\right) - i\lambda \int \frac{g_2(z) + g_3(z)}{f_1(z)} dz - i\lambda \ln(u^{*-1}v^{-1}) + ig_3(z) + \right. \\
&\quad \left. \delta(f_0(z) - \beta_0) + i\frac{\alpha_1}{2} z + \frac{\delta^2}{2} f_1(z)\right\},
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\Omega}_1 &\equiv (vv^*)^{-1} \exp\left(\int \frac{g_3(z) + g_4(z)}{f_1(z)} dz\right), \\
\tilde{\Omega}_2 &\equiv (uu^*)^{-1} \exp\left(\int \frac{g_1(z) + g_2(z)}{f_1(z)} dz\right).
\end{aligned}$$

Here $\tilde{\Psi}_1$ and $\tilde{\Psi}_2$ are arbitrary complex-valued functions. Note that $f_0$ and $f_1$ are given in Lemma 1.

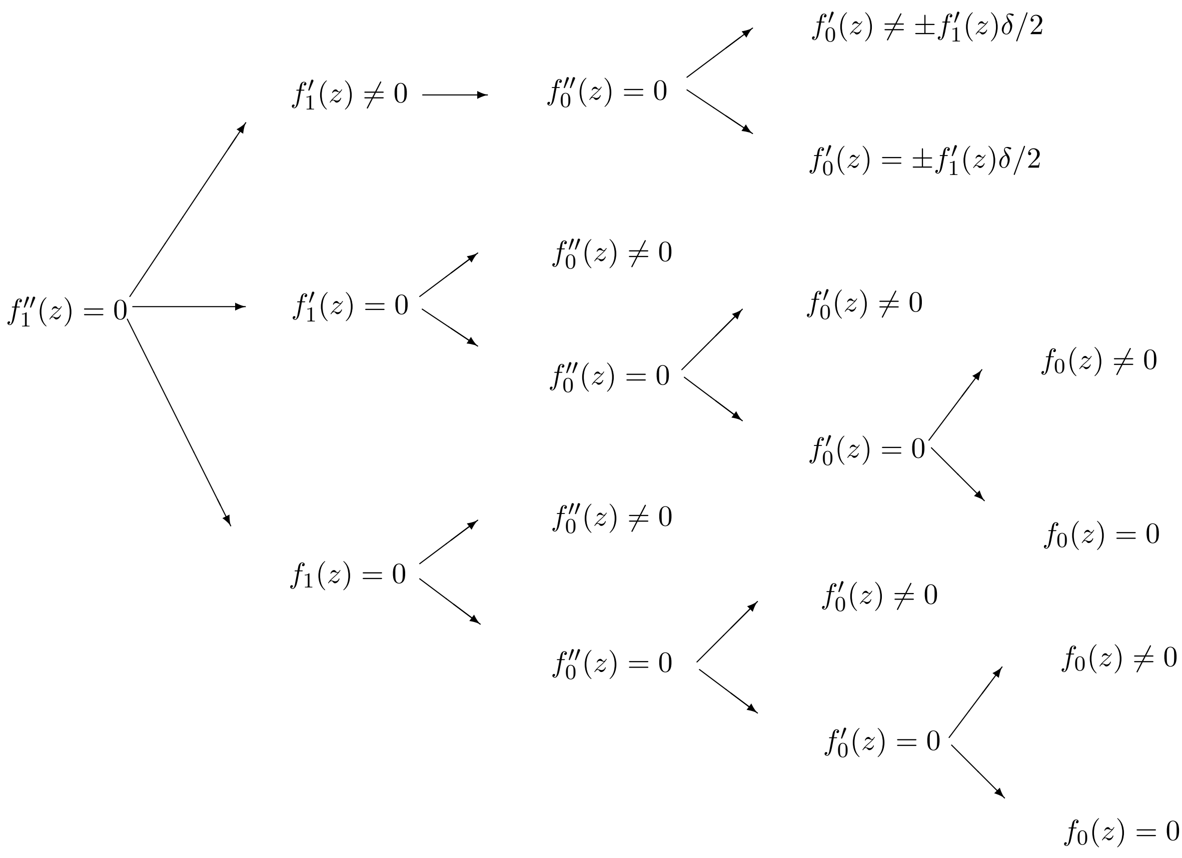


Figure 1: Subcases for $f_1''(z) = 0$

**Case 1.2** Let $a_j \neq 0;\ j = 1\ldots,4$. By solving the system (A-1), (A-2), and (A-6) we find that

$$\begin{aligned}
&\lambda = 0,\\
&a_1(z) = a_2^*(z), \quad a_3(z) = a_4^*(z),\\
&\frac{a_1(z)}{a_4(z)} = \frac{a_2(z)}{a_3(z)},\\
&a_1(z)a_3(z) = c_1, \quad a_2(z)a_4(z) = c_1, \quad c_1 \in \mathcal{R},\\
&\frac{a_1(z)}{a_4(z)} \exp\left(\frac{g_3(z)+g_4(z)}{f_1(z)}\right) \exp\left(-\frac{g_1(z)+g_2(z)}{f_1(z)}\right) = c_2, \quad c_2 \in \mathcal{R}.
\end{aligned}$$

The functions $F_1$ and $F_2$ take the following form:

$$F_1 = u\tilde{\Psi}_1(\tilde{\Omega}), \qquad F_2 = v\tilde{\Psi}_2(\tilde{\Omega}),$$

where

$$\tilde{\Omega} \equiv vv^* \exp\left(-\int \frac{g_3(z)+g_4(z)}{f_1(z)}dz\right) - c_2^{-1}uu^* \exp\left(-\int \frac{g_1(z)+g_2(z)}{f_1(z)}dz\right).$$

Here $\Psi_1$ and $\Psi_2$ are arbitrary complex-valued functions of $\tilde{\Omega}$.

**Case 2** Let $f_1''(z) = 0$. A diagram of the subcases is given in Figure 1. We shall prove the following

**Lemma 2** *If*

$$f_1''(z) = 0,$$

*$f_1$ and $f_0$ have the following general form:*

$$f_1(z) = \alpha_2 z + \alpha_3 \tag{20}$$

*and*
*i) for $f_1'(z) \neq 0$ and $f_0'(z) \neq \pm\delta f_1'(z)/2$,*

$$f_0 = \alpha_0 z + \beta_0, \tag{21}$$

*ii) for $f_1'(z) \neq 0$ and*
*a) $f_0'(z) = \delta f_1'(z)/2$,*

$$f_0 = \delta\alpha_2 z/2 + \beta_0, \tag{22}$$

*b) $f_0'(z) = -\delta f_1'(z)/2$,*

$$f_0 = -\delta\alpha_2 z/2 + \beta_0,, \tag{23}$$

*iii) for* $f_1'(z) = 0$ *and* $f_0''(z) \neq 0$,

$$f_0 = i\frac{\alpha_3}{\lambda_1} \exp\left(-i\frac{\lambda_1}{\alpha_3} z\right) + \beta_0, \tag{24}$$

*iv) for* $f_1'(z) = 0$, $f_0''(z) = 0$, *and* $f_0'(z) \neq 0$,

$$f_0 = \alpha_0 z + \beta_0, \quad (\alpha_0 \neq 0), \tag{25}$$

*v) for* $f_1'(z) = 0$ *(or* $f_1(z) = 0$*),* $f_0''(z) = 0$, $f_0'(z) = 0$,

$$f_0 = \beta_0, \tag{26}$$

*vi) for* $f_1(z) = 0$ *and* $f_0''(z) \neq 0$

$$f_0(z) = h(z), \qquad h''(z) \neq 0, \tag{27}$$

*vii) for* $f_1(z) = 0$, $f_0''(z) = 0$, *and* $f_0'(z) \neq 0$,

$$f_0 = \alpha_0 z + \beta_0. \tag{28}$$

*Here* $\alpha_2, \alpha_3, \beta_0, \lambda_1 \in \mathcal{R}$, *and* $h$ *is an arbitrary real function.*

**P r o o f** We prove only the assertion (i). From (A-1)–(A-5) it follows that

$$i(\alpha_2 z + \alpha_3)\frac{f_0''(z)}{f_0'(z) - \delta\alpha_2/2} = \lambda_1, \tag{29}$$

$$i(\alpha_2 z + \alpha_3)\frac{f_0''(z)}{f_0'(z) + \delta\alpha_2/2} = \tilde{\lambda}_1, \tag{30}$$

where $\lambda_1$ and $\tilde{\lambda}_1$ are arbitrary constants. Now, if $\lambda_1 = 0$ the general form of $f_0$ is

$$f_0(z) = \alpha_0 z + \beta_0,$$

so that $\tilde{\lambda}_1 = 0$. However, if $\lambda_1 \neq 0$ the differential equations (29) and (30) are not compatible. We thus conclude that (21) is the general form of $f_0$ for (i) in Lemma 2. □
The functional form of $F_1$ and $F_2$ in (1) can now be related to the Lie symmetry generators of (1) by

**Proposition 2** *The system (1) possesses the Lie symmetry generators (2) with infinitesimals (6)–(11) and*

*i)* $f_1$, $f_0$ *is given by (20) and (21) if and only if* $F_1$ *and* $F_2$ *take on the form*

$$F_1 = u\Psi_1(\Omega_1, \Omega_2, \Omega_3, z), \quad F_2 = v\Psi_2(\Omega_1, \Omega_2, \Omega_3, z),$$

*with*

$$\Omega_1 \equiv (uu^*)^{-1}, \quad \Omega_2 \equiv uv^{\Gamma}, \quad \Omega_3 \equiv (vv^*)^{-1},$$
$$\Gamma \equiv \frac{\delta\alpha_2 - 2\alpha_0}{\delta\alpha_2 + 2\alpha_0},$$

*and satisfy (A-1), (A-2), and (A-7), or*

*ii)* $f_1$, $f_0$ *is given by (20) and*
*a) (22) if and only if* $F_1$ *and* $F_2$ *take on the form*

$$F_1 = \Psi_1(u, u^*, \Omega_3, z), \quad F_2 = v\Psi_2(u, u^*, \Omega_3, z),$$

*with*

$$\Omega_3 \equiv (vv^*)^{-1},$$

*and satisfy (A-1), (A-2), and (A-8), or*
*b) (23) if and only if* $F_1$ *and* $F_2$ *take on the form*

$$F_1 = u\Psi_1(\Omega_1, v, v^*, z), \quad F_2 = \Psi_2(\Omega_1, v, v^*, z)$$

*with*

$$\Omega_1 \equiv (uu^*)^{-1},$$

*and satisfy (A-1), (A-2), and (A-9), or*

*iii)* $f_1$, $f_0$ *is given by (20) (with* $\alpha_2 = 0$*) and (24) if and only if* $F_1$ *and* $F_2$ *take on the form*

$$F_1 = u\Psi_1(\Omega_1, \Omega_2, \Omega_3, z) + \frac{\lambda_1}{\alpha_3} u \ln u, \quad F_2 = v\Psi_2(\Omega_1, \Omega_2, \Omega_3, z) + \frac{\lambda_1}{\alpha_3} v \ln v,$$

*with*

$$\Omega_1 \equiv (uu^*)^{-1}, \quad \Omega_2 \equiv uv^{-1}, \quad \Omega_3 \equiv (vv^*)^{-1},$$

*and satisfy (A-1), (A-2), and (A-10), or*

*iv)* $f_1$, $f_0$ *is given by (20) (with* $\alpha_2 = 0$*) and (25) if and only if* $F_1$ *and* $F_2$ *take on the form*

$$F_1 = u\Psi_1(\Omega_1, \Omega_2, \Omega_3), \quad F_2 = v\Psi_2(\Omega_1, \Omega_2, \Omega_3),$$

*with*

$$\Omega_1 \equiv (uu^*)^{-1}, \quad \Omega_2 \equiv uv^{-1}, \quad \Omega_3 \equiv (vv^*)^{-1},$$

*and satisfy (A-1), (A-2), and (A-11), or*

*v)* $f_1$, $f_0$ *is given by (20) (with* $\alpha_2 = 0$ *and* $\alpha_3 \neq 0$ *or* $\alpha_2 = 0$ *and* $\alpha_3 = 0$*) and (26) if and only if* $F_1$ *and* $F_2$ *take on the form*

$$F_1 = \Psi_1(u, u^*, v, v^*, z), \quad F_2 = \Psi_2(u, u^*, v, v^*, z)$$

*and satisfy (A-1), (A-2), and (A-5), or* *vi) $f_1$, $f_0$ is given by (20) (with $\alpha_2 = 0$ and $\alpha_3 = 0$) and (27) if and only if $F_1$ and $F_2$ take on the form*

$$F_1 = i\frac{f_0''(z)}{f'(z)} u \ln u + u\Psi_1(\Omega_1, \Omega_2, \Omega_3, z),$$
$$F_2 = i\frac{f_0''(z)}{f'(z)} v \ln v + v\Psi_2(\Omega_1, \Omega_2, \Omega_3, z)$$

*with*

$$\Omega_1 \equiv (uu^*)^{-1}, \quad \Omega_2 \equiv uv^{-1}, \quad \Omega_3 \equiv (vv^*)^{-1},$$

*and satisfy (A-1), (A-2), and (A-12), or*

*vii) $f_1$, $f_0$ is given by (20) (with $\alpha_2 = 0$ and $\alpha_3 = 0$) and (28) if and only if $F_1$ and $F_2$ take on the form*

$$F_1 = u\Psi_1(\Omega_1, \Omega_2, \Omega_3, z), \quad F_2 = v\Psi_2(\Omega_1, \Omega_2, \Omega_3, z),$$

*with*

$$\Omega_1 \equiv (uu^*)^{-1}, \quad \Omega_2 \equiv uv^{-1}, \quad \Omega_3 = (vv^*)^{-1},$$

*and satisfy (A-1), (A-2), (A-11) with $\alpha_3 = 0$.*

*Here $\Psi_1$ and $\Psi_2$ are arbitrary complex-valued functions.*

For solving systems (A-1), (A-2), and (A-7) – (A-12) we have to distinguish between different values of $g_j$; $j = 1, \ldots, 4$. We now consider some examples:

**Case 2.1** Let

$$g_1(z) + g_2(z) \neq 0, \quad g_1(z) + \Gamma g_3(z) \neq 0, \quad g_3(z) + g_4(z) \neq 0,$$

where $g_1(z) = g_2^*(z)$ and $g_3(z) = g_4^*(z)$. With the above assumptions we solve the system (A-1), (A-2), and (A-7). This results in two subcases:

**Subcase 2.1.1** **(i)** At first we consider $a_j = 0$; $j = 1, \ldots, 4$. From (i) in Lemma 2 it follows that

$$\begin{aligned} F_1 &= \frac{u}{\alpha_2 z + \alpha_3}\left\{\tilde{\Psi}_1(\tilde{\Omega}_1, \tilde{\Omega}_2, \tilde{\Omega}_3) - \delta\left(\alpha_0 - \frac{\delta}{2}\alpha_2\right) z + i g_1(z)\right\}, \\ F_2 &= \frac{v}{\alpha_2 z + \alpha_3}\left\{\tilde{\Psi}_2(\tilde{\Omega}_1, \tilde{\Omega}_2, \tilde{\Omega}_3) + \delta\left(\alpha_0 + \frac{\delta}{2}\alpha_2\right) z + i g_3(z)\right\}, \end{aligned}$$

where $\tilde{\Psi}_1$, $\tilde{\Psi}_2$ are arbitrary complex functions, and

$$\tilde{\Omega}_1 \equiv (uu^*)^{-1} \exp\left(\int \frac{g_1(z) + g_2(z)}{\alpha_2 z + \alpha_3} dz\right), \quad \tilde{\Omega}_2 \equiv uv^{\Gamma} \exp\left(-\int \frac{g_1(z) + \Gamma g_3(z)}{\alpha_2 z + \alpha_3} dz\right),$$
$$\tilde{\Omega}_3 \equiv (vv^*)^{-1} \exp\left(\int \frac{g_3(z) + g_4(z)}{\alpha_2 z + \alpha_3} dz\right).$$

**Subcase 2.1.2 (i)** Here $a_j \neq 0;\ j = 1, \ldots, 4$. From (i) in Lemma 2 we obtain

$$\begin{aligned}
F_1 &= \frac{u}{\alpha_2 z + \alpha_3}\left\{\tilde{\Psi}_1(\tilde{\Omega}) - \delta\left(\alpha_0 - \frac{\delta}{2}\alpha_2\right) z + i g_1(z)\right\},\\
F_2 &= \frac{v}{\alpha_2 z + \alpha_3}\left\{\tilde{\Psi}_1(\tilde{\Omega}) + \delta\left(\alpha_0 + \frac{\delta}{2}\alpha_2\right) z + i g_1(z) - 2\delta\alpha_0 z - i\frac{a_1'(z)}{a_1(z)}(\alpha_2 z + \alpha_3)\right\},
\end{aligned}$$

where $\tilde{\Psi}_1$ is an arbitrary complex function, and

$$\tilde{\Omega} \equiv c_1 v v^* \exp\left(-\int \frac{g_3(z) + g_4(z)}{\alpha_2 z + \alpha_3} dz\right) - u u^* \exp\left(-\int \frac{g_1(z) + g_2(z)}{\alpha_2 z + \alpha_3} dz\right),$$

$$\frac{a_1(z)}{a_4(z)} = \frac{a_2(z)}{a_3(z)} \equiv c_1 \in \mathcal{R},$$

$$\exp\left(-\int \frac{g_1(z) + g_2(z)}{\alpha_2 z + \alpha_3} dz\right) \exp\left(\int \frac{g_3(z) + g_4(z)}{\alpha_2 z + \alpha_3} dz\right) = 1.$$

If $c_1$ is not a constant, then

$$\begin{aligned}
F_1 &= \frac{u}{\alpha_2 z + \alpha_3}\left\{\tilde{c}_1 - \delta\left(\alpha_0 - \frac{\delta}{2}\alpha_2\right) z + i g_1(z)\right\},\\
F_2 &= \frac{v}{\alpha_2 z + \alpha_3}\left\{\tilde{c}_2 + \delta\left(\alpha_0 + \frac{\delta}{2}\alpha_2\right) z + i g_3(z)\right\},
\end{aligned}$$

where $\tilde{c}_1$ and $\tilde{c}_2$ are artbitrary complex constants, and

$$a_1(z) = \exp\left[2i\delta\frac{\alpha_0 z}{\alpha_2} - \int \frac{g_3(z) - g_1(z)}{\alpha_2 z + \alpha_3} dz\right] (\alpha_2 z + \alpha_3)^{-2i\delta\alpha_0\alpha_3/\alpha_2^2 + i(\tilde{c}_2 - \tilde{c}_1)/\alpha_2} \times \alpha_2^{2i\delta\alpha_0\alpha_3/\alpha_2^2},$$

$$a_3(z) = \exp\left[-2i\delta\frac{\alpha_0 z}{\alpha_2} + \int \frac{g_3(z) - g_1(z)}{(\alpha_2 z + \alpha_3} dz\right] (\alpha_2 z + \alpha_3)^{2i\delta\alpha_0\alpha_3/\alpha_2^2 - i(\tilde{c}_2 - \tilde{c}_1)/\alpha_2} \times \alpha_2^{-2i\delta\alpha_0\alpha_3/\alpha_2^2},$$

$$a_1(z) = a_2^*(z), \quad a_3(z) = a_4^*(z).$$

**Case 2.2** Let

$$g_1(z) \neq 0, \quad g_2(z) \neq 0, \quad g_3(z) + g_4(z) \neq 0, \quad g_2(z) - \alpha_2 \neq 0, \quad g_1(z) - \alpha_2 \neq 0,$$

where $g_1(z) = g_2^*(z)$ and $g_3(z) = g_4^*(z)$. With the above assumptions we solve the system (A-1), (A-2), and (A-8). Let us only consider the case (ii)(a) of Lemma 2. This results in the following subcases:

**Subcase 2.2.1 (ii)** At first we consider $a_j = 0;\ j = 1, \ldots, 4$. From (ii)(a) in Lemma 2 it follows that

$$\begin{aligned}
F_1 &= \frac{1}{\alpha_2 + \alpha_3}\left\{\tilde{\Psi}_1(\tilde{\Omega}_1, \tilde{\Omega}_2, \tilde{\Omega}_3) \exp\left(\int \frac{g_1(z)}{\alpha_2 z + \alpha_3} dz\right) + i g_1(z) u\right\},\\
F_2 &= \frac{v}{\alpha_2 z + \alpha_3}\left\{\tilde{\Psi}_2(\tilde{\Omega}_1, \tilde{\Omega}_2, \tilde{\Omega}_3) + i g_3(z) + \delta^2\alpha_2 z\right\},
\end{aligned}$$

where $\tilde{\Psi}_1$ and $\tilde{\Psi}_2$ are arbitrary complex-valued functions, and

$$\tilde{\Omega}_1 \equiv u \exp\left(-\int \frac{g_1(z)}{\alpha_2 z + \alpha_3} dz\right), \quad \tilde{\Omega}_2 \equiv u^* \exp\left(-\int \frac{g_2(z)}{\alpha_2 z + \alpha_3} dz\right),$$
$$\tilde{\Omega}_3 \equiv (vv^*)^{-1} \exp\left(\int \frac{g_3(z) + g_4(z)}{\alpha_2 z + \alpha_3} dz\right).$$

**Subcase 2.2.2 (ii)** Here we let $a_j \neq 0;\ j = 1, \ldots, 4$. From (ii)(a) in Lemma 2 it then follows that

$$F_1 = iu \frac{g_1(z)}{\alpha_2 z + \alpha_3}$$
$$F_3 = \frac{v}{\alpha_2 z + \alpha_3} \left\{ i g_3(z) + \delta^2 \alpha_2 z + c_1 \right\},$$

where $c_1$ is an arbitrary complex constant and

$$a_3(z) a_1(z) \equiv c_2, \quad c_2 \in \mathcal{C},$$
$$a_1^*(z) = a_2^*(z), \quad a_3^*(z) = a_4^*(z),$$
$$a_1(z) = (\alpha_2 z + \alpha_3)^{i(c_1 - \delta^2 \alpha_3)/\alpha_2} \exp\left(\int \frac{g_1(z) - g_3(z)}{\alpha_2 z + \alpha_3} dz + i\delta^2 z\right).$$

Let us choose two more examples which are of interest to us, namely the cases where the Galilean transformation is included, i.e., (iv) and (viii) in Lemma 2.

**Case 2.3** Let

$$g_1(z) = -i\delta(\alpha_0 z + \beta_0) + \gamma_1, \quad g_3(z) = i\delta(\alpha_0 z + \beta_0) + \gamma_2,$$

where $g_1(z) = g_2^*(z)$ and $g_3(z) = g_4^*(z)$, and $\gamma_1, \gamma_2 \in \mathcal{C}$. With the above assumptions we solve the system (A-1), (A-2), and (A-11) for the case (iv) in Lemma 2. We consider only $a_j = 0;\ j = 1, \ldots, 4$. It follows that

$$F_1 = u\tilde{\Psi}_1(\tilde{\Omega}_1, \tilde{\Omega}_2, \tilde{\Omega}_3), \quad F_2 = v\tilde{\Psi}_2(\tilde{\Omega}_1, \tilde{\Omega}_2, \tilde{\Omega}_3)$$

where $\tilde{\Psi}_1$ and $\tilde{\Psi}_2$ are arbitrary complex-valued functions, and

$$\tilde{\Omega}_1 \equiv (uu^*)^{-1} \exp\left(\frac{\gamma_1 + \gamma_1^*}{\alpha_3} z\right), \quad \tilde{\Omega}_2 \equiv (vv^*)^{-1} \exp\left(\frac{\gamma_2 + \gamma_2^*}{\alpha_3} z\right),$$
$$\tilde{\Omega}_3 \equiv uv^{-1} \exp\left(-i\delta \frac{\alpha_0}{\alpha_3} z^2 - 2i\delta \frac{\beta_0}{\alpha_3} z - \frac{\gamma_1 - \gamma_2}{\alpha_3} z\right).$$

Here $\alpha_0 \neq 0$ and $\alpha_3 \neq 0$.

**Case 2.4** Let

$$g_1(z) + g_2(z) \neq 0, \quad g_3(z) - g_1(z) \neq 0, \quad g_3(z) + g_4(z) \neq 0,$$
$$i g_1'(z) - \delta f_0'(z) \neq 0, \quad i g_3'(z) - \delta f_0'(z) \neq 0,$$

where $g_1^*(z) = g_2(z)$, and $g_3^*(z) = g_4(z)$. With this assumption we solve the system (A-1), (A-2) and (A-11) for the case (vii) of Lemma 2. We consider $a_j = 0$; $j = 1 \ldots, 4$. The functions $F_1$ and $F_2$ in (1) then take the form

$$\begin{aligned}
F_1 &= \frac{\delta\alpha_0 - ig_1'(z)}{g_1(z)+g_2(z)} u \left\{\tilde{\Psi}_1(\tilde{\Omega}_1, \tilde{\Omega}_2, z) + \ln \tilde{\Omega}_1\right\},\\
F_2 &= -\frac{\delta\alpha_0 - ig_3'(z)}{g_3(z)+g_4(z)} v \left\{\tilde{\Psi}_2(\tilde{\Omega}_1, \tilde{\Omega}_2, z) + \ln \tilde{\Omega}_1\right\},
\end{aligned}$$

where $\tilde{\Psi}_1$ and $\tilde{\Psi}_2$ are arbitrary complex-valued functions and

$$\tilde{\Omega}_1 \equiv (uu^*)^{-1}\left\{uv^{-1}\right\}^{-(g_1+g_2)/(g_3-g_1)}, \quad \tilde{\Omega}_2 \equiv (uu^*)^{-1}\left\{(vv^*)^{-1}\right\}^{-(g_1+g_2)/(g_3+g_4)}.$$

Let us now examine some special forms of (1) which are of importance in study of birefringent optical fibres.

**Case 3** Consider [5–10]

$$\begin{aligned}
&i\frac{\partial u}{\partial z} + i\delta\frac{\partial u}{\partial \tau} + \frac{1}{2}\frac{\partial^2 u}{\partial \tau^2} - i\gamma u + \left(|u|^2 + \varepsilon|v|^2\right)u = 0,\\
&i\frac{\partial v}{\partial z} - i\delta\frac{\partial v}{\partial \tau} + \frac{1}{2}\frac{\partial^2 v}{\partial \tau^2} - i\gamma v + \left(\varepsilon|u|^2 + |v|^2\right)v = 0,
\end{aligned} \tag{31}$$

where $\gamma, \varepsilon \in \mathcal{R}$. The Lie symmetry generators are given by

**Case 3.1** For $\varepsilon$ arbitrary:

$$< G_1,\ G_2,\ G_5,\ G_6,\ G_8 > .$$

**Case 3.2** For $\varepsilon = 1$:

$$< G_1,\ G_2,\ G_3,\ G_3',\ G_4,\ G_4',\ G_5,\ G_5',\ G_6,\ G_6',\ G_8 > .$$

Note that the definitions of $G_1$, etc. are given in Theorem 1.

**Case 4** Consider

$$\begin{aligned}
&i\frac{\partial u}{\partial z} + i\delta\frac{\partial u}{\partial \tau} + \frac{1}{2}\frac{\partial^2 u}{\partial \tau^2} - i\gamma u + \left(|u|^2 + \varepsilon|v|^2\right)u + v^2u^*h_1(z) = 0,\\
&i\frac{\partial v}{\partial z} - i\delta\frac{\partial v}{\partial \tau} + \frac{1}{2}\frac{\partial^2 v}{\partial \tau^2} - i\gamma v + \left(\varepsilon|u|^2 + |v|^2\right)v + u^2v^*h_2(z) = 0,
\end{aligned} \tag{32}$$

where $\gamma, \varepsilon \in \mathcal{R}$. The only functions $h_1$ and $h_2$ which admit a Galilean generator are

$$\begin{aligned}
h_1(z) &= \alpha_1 \exp\left[2(-i\delta k_0 z^2 - i\beta_1 z)\right],\\
h_2(z) &= \alpha_2 \exp\left[2(i\delta k_0 z^2 + i\beta_1 z)\right],
\end{aligned}$$

where the real constants $k_i, \alpha_i, \beta_j$ $(i = 1, 2;\ j = 1, 2, 3)$ are connected to the infinitesimal functions $\xi_0, \xi_1, \eta_j;\ j = 1, \ldots, 4$, of the Lie symmetries in the following way:

$$\begin{aligned}
\xi_0 &= k_0 z + k_1, \qquad \xi_1 = 1,\\
\eta_1 &= (ik_0\tau - ik_0\delta z + i\beta_2)\, u,\\
\eta_2 &= (-ik_0\tau + ik_0\delta z - i\beta_2)\, u^*,\\
\eta_3 &= (ik_0\tau + ik_0\delta z + i\beta_3)\, v,\\
\eta_4 &= (-ik_0\tau - ik_0\delta z - i\beta_3)\, v^*
\end{aligned}$$

with the condition

$$\beta_1 + \beta_2 - \beta_3 = 0.$$

For $k_0 = 0$ we obtain the system proposed by Menyuk [7]. Note that for $k_0 \neq 0$ a Galilean invariant system is obtained.

**Case 5** Finally we consider a system of Schrödinger equations which appear in study of single-mode optical fibres [11]:

$$\begin{aligned}
&i\frac{\partial u}{\partial z} + i\delta\frac{\partial u}{\partial \tau} + \frac{1}{2}\frac{\partial^2 u}{\partial \tau^2} + \left(|u|^2 + \varepsilon|v|^2\right) u + \frac{\varepsilon}{2} v^2 u^* \exp(-iR\delta z) = 0,\\
&i\frac{\partial v}{\partial z} - i\delta\frac{\partial v}{\partial \tau} + \frac{1}{2}\frac{\partial^2 v}{\partial \tau^2} + \left(\varepsilon|u|^2 + |v|^2\right) v + \frac{\varepsilon}{2} u^2 v^* \exp(iR\delta z) = 0.
\end{aligned} \tag{33}$$

The maximal Lie symmetry generators of (33) are

$$\begin{aligned}
&G_2 = \frac{\partial}{\partial \tau},\\
&G_1 + \beta_2 G_5 + \beta_3 G_6 = \frac{\partial}{\partial z} + i\beta_2 u\frac{\partial}{\partial u} - i\beta_2 u^*\frac{\partial}{\partial u^*} + i\beta_3 v\frac{\partial}{\partial v} - i\beta_3 v^*\frac{\partial}{\partial v^*},
\end{aligned} \tag{34}$$

where

$$\frac{1}{2}R\delta + \beta_2 - \beta_3 = 0.$$

The Lagrangian is given by

$$\begin{aligned}
\mathcal{L} \;=\; & \frac{i}{2}\left(u\frac{\partial u^*}{\partial z} - u^*\frac{\partial u}{\partial z}\right) + \frac{i}{2}\left(v\frac{\partial v^*}{\partial z} - v^*\frac{\partial v}{\partial z}\right) +\\
& \frac{i\delta}{2}\left(u\frac{\partial u^*}{\partial \tau} - u^*\frac{\partial u}{\partial \tau}\right) + \frac{i\delta}{2}\left(v^*\frac{\partial v}{\partial \tau} - v\frac{\partial v^*}{\partial \tau}\right) +\\
& \frac{1}{2}\left(\left|\frac{\partial u}{\partial \tau}\right|^2 - |u|^4\right) + \frac{1}{2}\left(\left|\frac{\partial v}{\partial \tau}\right|^2 - |v|^4\right) -\\
& \varepsilon|u|^2|v|^2 - \frac{\varepsilon}{4}u^2(v^*)^2\exp(iR\delta z) - \frac{\varepsilon}{4}(u^*)^2 v^2 \exp(-iR\delta z).
\end{aligned} \tag{35}$$

By using the linear combination

$$G_2 + k(G_1 + \beta_2 G_5 + \beta_3 G_6)$$

the symmetry ansatz follows from the first integrals of the autonomous system

$$\begin{aligned} &\frac{dz}{d\tilde{\varepsilon}} = 1, \qquad && \frac{d\tau}{d\tilde{\varepsilon}} = k, \\ &\frac{du}{d\tilde{\varepsilon}} = i\beta_2 u, && \frac{du^*}{d\tilde{\varepsilon}} = -i\beta_2 u^*, \\ &\frac{dv}{d\tilde{\varepsilon}} = i\beta_2 v, && \frac{dv^*}{d\tilde{\varepsilon}} = i\beta_3 v^*, \end{aligned}$$

i.e.,

$$\begin{aligned} &u(z,\tau) = \varphi_1(\omega(z,\tau))\exp(i\beta_2 z), \\ &v(z,\tau) = \varphi_2(\omega(z,\tau))\exp(i\beta_3 z), \\ &\omega \equiv \tau - kz. \end{aligned} \tag{36}$$

Here $\varphi_1$, $\varphi_2$ are the new complex-valued dependent variables, and $\omega$ is the new independent variable. With (36) system (33) reduces to the following coupled ordinary differential equations (ODE's):

$$\frac{1}{2}\ddot{\varphi}_1 + i(\delta - k)\dot{\varphi}_1 - \beta_2\varphi_1 + (|\varphi_1|^2 + \varepsilon|\varphi_2|^2)\varphi_1 + \frac{\varepsilon}{2}\varphi_2^2\varphi_1^* = 0, \tag{37}$$

$$\frac{1}{2}\ddot{\varphi}_2 - i(\delta + k)\dot{\varphi}_2 - \beta_3\varphi_2 + (|\varphi_2|^2 + \varepsilon|\varphi_1|^2)\varphi_2 + \frac{\varepsilon}{2}\varphi_1^2\varphi_2^* = 0, \tag{38}$$

whereas the Lagrangian (35) reduces to

$$\begin{aligned} \mathcal{L} \;=\; & \frac{1}{2}\left(|\dot{\varphi}_1|^2 - |\varphi_1|^4\right) + \frac{1}{2}\left(|\dot{\varphi}_2|^2 - |\varphi_2|^4\right) + \beta_2\varphi_1^2 + \beta_3\varphi_2^2 - \\ & \varepsilon|\varphi_1|^2|\varphi_2|^2 - \frac{\varepsilon}{4}\left(\varphi_1^2(\varphi_2^*)^2 - (\varphi_1^*)^2\varphi_2^2\right). \end{aligned}$$

Here $\dot{\varphi}_1 \equiv d\varphi_1/d\omega$, etc. The maximal Lie symmetries of the system (37), (38) are

$$\frac{\partial}{\partial\omega}, \quad i\varphi\frac{\partial}{\partial\varphi} - i\varphi^*\frac{\partial}{\partial\varphi^*} + i\Psi\frac{\partial}{\partial\Psi} - i\Psi^*\frac{\partial}{\partial\Psi^*}.$$

For $\varepsilon = 2/3$ equation (33) can be transformed by the change of variables

$$\begin{aligned} A(z,\tau) &= \left(\frac{1}{3}\right)^{1/2}\left[u\exp\left(i\frac{R\delta}{4}z\right) + iv\exp\left(-i\frac{R\delta}{4}z\right)\right], \\ B(z,\tau) &= \left(\frac{1}{3}\right)^{1/2}\left[u\exp\left(i\frac{R\delta}{4}z\right) - iv\exp\left(-i\frac{R\delta}{4}z\right)\right], \end{aligned}$$

thus it takes the form:

$$\begin{aligned} &i\frac{\partial A}{\partial z} + i\delta\frac{\partial B}{\partial \tau} + \frac{1}{2}\frac{\partial^2 A}{\partial\tau^2} + \frac{R\delta}{4}B + (|A|^2 + 2|B|^2)A = 0, \\ &i\frac{\partial B}{\partial z} + i\delta\frac{\partial A}{\partial \tau} + \frac{1}{2}\frac{\partial^2 B}{\partial\tau^2} + \frac{R\delta}{4}A + (|B|^2 + 2|A|^2)B = 0. \end{aligned} \tag{39}$$

The Lagrangian of the system (39) is

$$\begin{aligned}\mathcal{L} \;=\; & \frac{3}{4}\Bigg\{2i\frac{R\delta}{4}\left(AB^* + BA^* + \frac{\partial A^*}{\partial z}A + \frac{\partial B^*}{\partial z}B\right) - \left(\frac{\partial A}{\partial z}A^* + \frac{\partial B}{\partial z}B^*\right) + \\ & i\delta\left(\frac{\partial A^*}{\partial \tau}B + \frac{\partial B^*}{\partial \tau}A - \frac{\partial A}{\partial \tau}B^* - \frac{\partial B}{\partial \tau}A^*\right) + \\ & \left|\frac{\partial A}{\partial \tau}\right|^2 + \left|\frac{\partial B}{\partial \tau}\right|^2 - |A|^4 - |B|^4 - 4|A|^2|B|^2\Bigg\}.\end{aligned}$$

The maximal Lie symmetries of (39) (with $\delta \neq 0$) are the following:

$$G_1 = \frac{\partial}{\partial z}, \qquad G_2 = \frac{\partial}{\partial \tau}, \qquad \tilde{G}_3 = iA\frac{\partial}{\partial A} - iA^*\frac{\partial}{\partial A^*} + iB\frac{\partial}{\partial B} - iB^*\frac{\partial}{\partial B^*}.$$

From the combination

$$G_1 + k_2 G_2 + k_3 \tilde{G}_3$$

the symmetry ansatz

$$A = \Phi(\omega)\exp(ik_3 z), \qquad B = \Psi(\omega)\exp(ik_3 z),$$
$$\omega = \tau - k_2 z$$

is obtained. The system (39) can then be reduced to

$$\frac{1}{2}\ddot{\Phi} - ik_2\dot{\Phi} + i\delta k_1\dot{\Psi} - k_3\Phi + \frac{R\delta}{4}\Psi + (|\Phi|^2 + 2|\Psi|^2)\Phi = 0, \tag{40}$$

$$\frac{1}{2}\ddot{\Psi} - ik_2\dot{\Psi} + i\delta k_1\dot{\Phi} - k_3\Psi + \frac{R\delta}{4}\Phi + (|\Psi|^2 + 2|\Phi|^2)\Psi = 0. \tag{41}$$

Let us finally make some remarks about applications of the symmetry properties of the system (1) which we will consider in a future paper:

1) Reduction to a system of ODE's. As we have demonstrated in the case 5, the Lie symmetry generators of (1) can be used to reduce (1) to systems of ODE's. Exact solutions of these ODE's can be constructed which in turn lead to exact solutions of (1). It is well-known that an exact solution of a PDE which is obtained from its symmetry generator defines, in fact, an infinite number of solutions; all invariant under that symmetry, i.e., the group parameter is arbitrary.

2) Canonical variables can be constructed for the reduced ODE's. The Lie generators of the reduced systems of ODE's can be used to define a set of new variables which then reduces the order of the system of ODE's.

3) Conservation laws. The Lie generators of (1) can be used to construct conservation laws for the system (1). When the system (1) has a Lagrangian $\mathcal{L}$ the conserved current $\zeta$ is given by

$$\zeta := \chi - G \rfloor \Theta.$$

This is obtained by calculating

$$L_G\Theta = d\chi,$$

i.e., the Lie derivative of the Cartan fundamental form $\Theta$ with respect to a symmetry generator $G$. The Cartan fundamental form in a space of $n$-dependent and $m$-independent variables is given by the $m$-form

$$\Theta = \left(\mathcal{L} - \sum_{j=1}^{n}\sum_{i=1}^{m}\frac{\partial \mathcal{L}}{\partial u_{j,i}}u_{j,i}\right)\Omega + \sum_{j=1}^{n}\sum_{i=1}^{m}\frac{\partial \mathcal{L}}{\partial u_{j,i}}du_j \wedge \left(\frac{\partial}{\partial x_i}\lrcorner\Omega\right),$$

where the volume $m$-form is indicated by $\Omega \equiv dx_1 \wedge \cdots \wedge dx_m$. Note that

$$L_G\Theta \equiv G\lrcorner d\Theta + d(G\lrcorner\Theta).$$

The Euler–Lagrange equation is given by

$$js^*(G\lrcorner d\Theta) = 0.$$

Note that, if $\zeta$ is a conserved current then $L_Z\zeta$ is also a conserved current. The conservation law is obtained from $d(js^*\zeta) = 0$. In the same way we can use the Lie generators of the reduced systems of ODE's to construct first integrals for these systems.

The problem of classification of the system (1) with respect to the Galilean group was solved in [12].

## Appendix A

The determining equations and conditions on $F_1$ and $F_2$ for the Lie symmetries of the system (1) are given by the following systems:

$$a_1(z)F_2 + ia_1'(z)v - a_4(z)u^*\frac{\partial F_1}{\partial v^*} - a_1(z)v\frac{\partial F_1}{\partial u} = 0, \qquad \text{(A-1.1)}$$

$$a_2(z)F_2^* - ia_2'(z)v^* - a_2(z)v^*\frac{\partial F_1^*}{\partial u^*} - a_3(z)u\frac{\partial F_1^*}{\partial v} = 0, \qquad \text{(A-1.2)}$$

$$a_3(z)F_1 + ia_3'(z)u - a_3(z)u\frac{\partial F_2}{\partial v} - a_2(z)v^*\frac{\partial F_2}{\partial u^*} = 0, \qquad \text{(A-1.3)}$$

$$a_4(z)F_1^* - ia_4'(z)u^* - a_4(z)u^*\frac{\partial F_2^*}{\partial v^*} - a_1(z)v\frac{\partial F_2^*}{\partial u} = 0; \qquad \text{(A-1.4)}$$

$$v^*a_2(z)\frac{\partial F_1}{\partial u^*} + a_3(z)u\frac{\partial F_1}{\partial v} = 0, \qquad \text{(A-2.1)}$$

$$va_1(z)\frac{\partial F_1^*}{\partial u} + a_4(z)u^*\frac{\partial F_1^*}{\partial v^*} = 0, \qquad \text{(A-2.2)}$$

$$va_1(z)\frac{\partial F_2}{\partial u} + a_4(z)u^*\frac{\partial F_2}{\partial v^*} = 0, \qquad \text{(A-2.3)}$$

$$v^*a_2(z)\frac{\partial F_2^*}{\partial u^*} + a_3(z)u\frac{\partial F_2^*}{\partial v} = 0; \qquad \text{(A-2.4)}$$

$$f_1''(z)\left\{F_1 - u\frac{\partial F_1}{\partial u} + u^*\frac{\partial F_1}{\partial u^*} - v\frac{\partial F_1}{\partial v} + v^*\frac{\partial F_1}{\partial v^*}\right\} + iuf_1'''(z) = 0, \tag{A-3.1}$$

$$f_1''(z)\left\{-F_1^* + u^*\frac{\partial F_1^*}{\partial u^*} - u\frac{\partial F_1^*}{\partial u} - v\frac{\partial F_1^*}{\partial v} + v^*\frac{\partial F_1^*}{\partial v^*}\right\} + iu^*f_1'''(z) = 0, \tag{A-3.2}$$

$$f_1''(z)\left\{F_2 - v\frac{\partial F_2}{\partial v} - u\frac{\partial F_2}{\partial u} + u^*\frac{\partial F_2}{\partial u^*} + v^*\frac{\partial F_2}{\partial v^*}\right\} + ivf_1'''(z) = 0, \tag{A-3.3}$$

$$f_1''(z)\left\{-F_2^* + v^*\frac{\partial F_2^*}{\partial v^*} - u\frac{\partial F_2^*}{\partial u} + u^*\frac{\partial F_2^*}{\partial u^*} - v\frac{\partial F_2^*}{\partial v}\right\} + iv^*f_1'''(z) = 0; \tag{A-3.4}$$

$$\begin{aligned}&\left(f_0'(z) - \frac{\delta}{2}f_1'(z)\right)\left\{F_1 - u\frac{\partial F_1}{\partial u} + u^*\frac{\partial F_1}{\partial u^*}\right\} - \\ &\quad\left(f_0'(z) + \frac{\delta}{2}f_1'(z)\right)\left\{v\frac{\partial F_1}{\partial v} - v^*\frac{\partial F_1}{\partial v^*}\right\} + iuf_0'' = 0,\end{aligned} \tag{A-4.1}$$

$$\begin{aligned}&\left(f_0'(z) - \frac{\delta}{2}f_1'(z)\right)\left\{F_1^* - u^*\frac{\partial F_1^*}{\partial u^*} + u\frac{\partial F_1^*}{\partial u}\right\} + \\ &\quad\left(f_0'(z) + \frac{\delta}{2}f_1'(z)\right)\left\{v\frac{\partial F_1^*}{\partial v} - v^*\frac{\partial F_1^*}{\partial v^*}\right\} - iu^*f_0''(z) = 0,\end{aligned} \tag{A-4.2}$$

$$\begin{aligned}&\left(f_0'(z) + \frac{\delta}{2}f_1'(z)\right)\left\{F_2 - v\frac{\partial F_2}{\partial v} + v^*\frac{\partial F_2}{\partial v^*}\right\} - \\ &\quad\left(f_0'(z) - \frac{\delta}{2}f_1'(z)\right)\left\{u\frac{\partial F_2}{\partial u} - u^*\frac{\partial F_2}{\partial u^*}\right\} + ivf_0''(z) = 0,\end{aligned} \tag{A-4.3}$$

$$\begin{aligned}&\left(f_0'(z) + \frac{\delta}{2}f_1'(z)\right)\left\{F_2^* - v^*\frac{\partial F_2^*}{\partial v^*} + v\frac{\partial F_2^*}{\partial v}\right\} + \\ &\quad\left(f_0'(z) - \frac{\delta}{2}f_1'(z)\right)\left\{u\frac{\partial F_2^*}{\partial u} - u^*\frac{\partial F_2^*}{\partial u^*}\right\} - iv^*f_0''(z) = 0;\end{aligned} \tag{A-4.4}$$

$$\begin{aligned}&g_1(z)\left(F_1 - u\frac{\partial F_1}{\partial u}\right) - g_2(z)u^*\frac{\partial F_1}{\partial u^*} - g_3(z)v\frac{\partial F_1}{\partial v} - g_4(z)v^*\frac{\partial F_1}{\partial v^*} - \\ &\quad f_1(z)\frac{\partial F_1}{\partial z} - F_1f_1'(z) + \frac{i}{4}uf_1''(z) + iug_1'(z) - \\ &\quad \delta u\left(f_0'(z) - \frac{\delta}{2}f_1'(z)\right) = 0,\end{aligned} \tag{A-5.1}$$

$$\begin{aligned}&g_2(z)\left(F_1^* - u^*\frac{\partial F_1^*}{\partial u^*}\right) - g_1(z)u\frac{\partial F_1^*}{\partial u} - g_3(z)v\frac{\partial F_1^*}{\partial v} - g_4(z)v^*\frac{\partial F_1^*}{\partial v^*} - \\ &\quad f_1(z)\frac{\partial F_1^*}{\partial z} - F_1^*f_1'(z) - \frac{i}{4}u^*f_1''(z) - iu^*g_2'(z) - \\ &\quad \delta u^*\left(f_0'(z) - \frac{\delta}{2}f_1'(z)\right) = 0,\end{aligned} \tag{A-5.2}$$

$$g_3(z)\left(F_2 - v\frac{\partial F_2}{\partial v}\right) - g_1(z)u\frac{\partial F_2}{\partial u} - g_2(z)u^*\frac{\partial F_2}{\partial u^*} - g_4(z)v^*\frac{\partial F_2}{\partial v^*} -$$

$$f_1(z)\frac{\partial F_2}{\partial z} - F_2 f_1'(z) + \frac{i}{4} v f_1''(z) + iv g_3'(z) +$$
$$\delta v\left(f_0'(z) + \frac{\delta}{2} f_1'(z)\right) = 0, \tag{A-5.3}$$
$$g_4(z)\left(F_2^* - v^*\frac{\partial F_2^*}{\partial v^*}\right) - g_1(z)u\frac{\partial F_2^*}{\partial u} - g_2(z)u^*\frac{\partial F_2^*}{\partial u^*} - g_3(z)v\frac{\partial F_2^*}{\partial v} -$$
$$f_1(z)\frac{\partial F_2^*}{\partial z} - F_2^* f_1'(z) - \frac{i}{4} v^* f_1''(z) - iv^* g_4'(z) +$$
$$\delta v^*\left(f_0'(z) + \frac{\delta}{2} f_1'(z)\right) = 0. \tag{A-5.4}$$

The conditions on the complex-valued functions $\Psi_1$ and $\Psi_2$ in Proposition 1 are given by the following system of equations:

$$(g_3+g_4)\Omega_1\frac{\partial \Psi_1}{\partial \Omega_1} + (g_1+g_2)\Omega_2\frac{\partial \Psi_1}{\partial \Omega_2} - f_1\frac{\partial \Psi_1}{\partial z} + i\frac{\lambda}{f_1}(g_1-g_3) -$$
$$f_1'(z)\Psi_1 + ig_1'(z) - \delta\left(f_0'(z) - \frac{\delta}{2}f_1'(z)\right) + i\frac{1}{4}f_1''(z) = 0, \tag{A-6.1}$$

$$(g_3+g_4)\Omega_1\frac{\partial \Psi_1^*}{\partial \Omega_1} + (g_1+g_2)\Omega_2\frac{\partial \Psi_1^*}{\partial \Omega_2} - f_1\frac{\partial \Psi_1^*}{\partial z} - i\frac{\lambda}{f_1}(g_2-g_4) -$$
$$f_1'(z)\Psi_1^* - ig_2'(z) - \delta\left(f_0'(z) - \frac{\delta}{2}f_1'(z)\right) - i\frac{1}{4}f_1''(z) = 0, \tag{A-6.2}$$

$$(g_3+g_4)\Omega_1\frac{\partial \Psi_2}{\partial \Omega_1} + (g_1+g_2)\Omega_2\frac{\partial \Psi_2}{\partial \Omega_2} - f_1\frac{\partial \Psi_2}{\partial z} - (g_3+g_2) -$$
$$\frac{f_1}{\lambda}\left\{g_3'(z) - i\delta\left(f_0'(z) + \frac{\delta}{2}f_1'(z)\right) + i\frac{1}{4}f_1''(z)\right\} = 0, \tag{A-6.3}$$

$$(g_3+g_4)\Omega_1\frac{\partial \Psi_2^*}{\partial \Omega_1} + (g_1+g_2)\Omega_2\frac{\partial \Psi_2^*}{\partial \Omega_2} - f_1\frac{\partial \Psi_2^*}{\partial z} - (g_1+g_4) +$$
$$\frac{f_1}{\lambda}\left\{g_4'(z) + i\delta\left(f_0'(z) + \frac{\delta}{2}f_1'(z)\right) - i\frac{1}{4}f_1''(z)\right\} = 0, \tag{A-6.4}$$

where $\Omega_1 \equiv (uu^*)^{-1}$, $\Omega_2 \equiv uv^{-1}$, and $\Omega_3 \equiv (vv^*)^{-1}$. The conditions on the complex-valued functions $\Psi_1$ and $\Psi_2$ in Proposition 2(i) are given by the following system of equations:

$$\Omega_1(g_1+g_2)\frac{\partial \Psi_1}{\partial \Omega_1} - \Omega_2(g_1+\Gamma g_3)\frac{\partial \Psi_1}{\partial \Omega_2} + \Omega_3(g_3+g_4)\frac{\partial \Psi_1}{\partial \Omega_3} - (\alpha_2 z+\alpha_3)\frac{\partial \Psi_1}{\partial z} -$$
$$\alpha_2\Psi_1 + ig_1'(z) - \delta(\alpha_0 - \delta\alpha_2/2) = 0, \tag{A-7.1}$$

$$\Omega_1(g_1+g_2)\frac{\partial \Psi_1^*}{\partial \Omega_1} - \Omega_2(g_1+\Gamma g_3)\frac{\partial \Psi_1^*}{\partial \Omega_2} + \Omega_3(g_3+g_4)\frac{\partial \Psi_1^*}{\partial \Omega_3} - (\alpha_2 z+\alpha_3)\frac{\partial \Psi_1^*}{\partial z} -$$
$$\alpha_2\Psi_1^* - ig_2'(z) - \delta(\alpha_0 - \delta\alpha_2/2) = 0, \tag{A-7.2}$$

$$\Omega_1(g_1+g_2)\frac{\partial \Psi_2}{\partial \Omega_1} - \Omega_2(g_1+\Gamma g_3)\frac{\partial \Psi_2}{\partial \Omega_2} + \Omega_3(g_3+g_4)\frac{\partial \Psi_2}{\partial \Omega_3} - (\alpha_2 z+\alpha_3)\frac{\partial \Psi_2}{\partial z} -$$

$$\alpha_2\Psi_2 + ig_3'(z) + \delta(\alpha_0 + \delta\alpha_2/2) = 0, \tag{A-7.3}$$

$$\Omega_1(g_1+g_2)\frac{\partial\Psi_2^*}{\partial\Omega_1} - \Omega_2(g_1+\Gamma g_3)\frac{\partial\Psi_2^*}{\partial\Omega_2} + \Omega_3(g_3+g_4)\frac{\partial\Psi_2^*}{\partial\Omega_3} - (\alpha_2 z+\alpha_3)\frac{\partial\Psi_2^*}{\partial z} - \alpha_2\Psi_2^* - ig_4'(z) + \delta(\alpha_0 + \delta\alpha_2/2) = 0, \tag{A-7.4}$$

where $\Omega_1 \equiv (uu^*)^{-1}$, $\Omega_2 \equiv uv^\Gamma$, and $\Omega_3 \equiv (vv^*)^{-1}$. The conditions on the complex-valued functions $\Psi_1$ and $\Psi_2$ in Proposition 2 (ii) (a) are given by the following system of equations:

$$-ug_1\frac{\partial\Psi_1}{\partial u} - u^*g_2\frac{\partial\Psi_1}{\partial u^*} + \Omega_3(g_3+g_4)\frac{\partial\Psi_1}{\partial\Omega_3} - (\alpha_2 z+\alpha_3)\frac{\partial\Psi_1}{\partial z} + (g_1-\alpha_2)\Psi_1 + iug_1' = 0, \tag{A-8.1}$$

$$-ug_1\frac{\partial\Psi_1^*}{\partial u} - u^*g_2\frac{\partial\Psi_1^*}{\partial u^*} + \Omega_3(g_3+g_4)\frac{\partial\Psi_1^*}{\partial\Omega_3} - (\alpha_2 z+\alpha_3)\frac{\partial\Psi_1^*}{\partial z} + (g_2-\alpha_2)\Psi_1^* - iu^*g_2' = 0 \tag{A-8.2}$$

$$-ug_1\frac{\partial\Psi_2}{\partial u} - u^*g_2\frac{\partial\Psi_2}{\partial u^*} + \Omega_3(g_3+g_4)\frac{\partial\Psi_2}{\partial\Omega_3} - (\alpha_2 z+\alpha_3)\frac{\partial\Psi_2}{\partial z} - \alpha_2\Psi_2 + ig_3' + \delta^2\alpha_2 = 0, \tag{A-8.3}$$

$$-ug_1\frac{\partial\Psi_2^*}{\partial u} - u^*g_2\frac{\partial\Psi_2^*}{\partial u^*} + \Omega_3(g_3+g_4)\frac{\partial\Psi_2^*}{\partial\Omega_3} - (\alpha_2 z+\alpha_3)\frac{\partial\Psi_2^*}{\partial z} - \alpha_2\Psi_2^* - ig_4' + \delta^2\alpha_2 = 0, \tag{A-8.4}$$

where $\Omega_3 \equiv (vv^*)^{-1}$. The conditions on the complex-valued functions $\Psi_1$ and $\Psi_2$ in Proposition 2 (ii) (b) are given by the following system of equations:

$$-vg_3\frac{\partial\Psi_1}{\partial v} - v^*g_4\frac{\partial\Psi_1}{\partial v^*} + \Omega_1(g_1+g_2)\frac{\partial\Psi_1}{\partial\Omega_1} - (\alpha_2 z+\alpha_3)\frac{\partial\Psi_1}{\partial z} - \alpha_2\Psi_1 + ig_1' + \delta^2\alpha_2 = 0, \tag{A-9.1}$$

$$-vg_3\frac{\partial\Psi_1^*}{\partial v} - v^*g_4\frac{\partial\Psi_1^*}{\partial v^*} + \Omega_1(g_1+g_2)\frac{\partial\Psi_1^*}{\partial\Omega_1} - (\alpha_2 z+\alpha_3)\frac{\partial\Psi_1^*}{\partial z} - \alpha_2\Psi_1^* - ig_2' + \delta^2\alpha_2 = 0, \tag{A-9.2}$$

$$-vg_3\frac{\partial\Psi_2}{\partial v} - v^*g_4\frac{\partial\Psi_2}{\partial v^*} + \Omega_1(g_1+g_2)\frac{\partial\Psi_2}{\partial\Omega_1} - (\alpha_2 z+\alpha_3)\frac{\partial\Psi_2}{\partial z} + (g_3-\alpha_2)\Psi_2 + ivg_3' = 0, \tag{A-9.3}$$

$$-vg_3\frac{\partial\Psi_2^*}{\partial v} - v^*g_4\frac{\partial\Psi_2^*}{\partial v^*} + \Omega_1(g_1+g_2)\frac{\partial\Psi_2^*}{\partial\Omega_1} -$$
$$(\alpha_2 z+\alpha_3)\frac{\partial\Psi_2^*}{\partial z} + (g_4-\alpha_2)\Psi_2^* - iv^*g_4' = 0, \quad \text{(A-9.4)}$$

where $\Omega_1 \equiv (uu^*)^{-1}$. The conditions on the complex-valued functions $\Psi_1$ and $\Psi_2$ in Proposition 2 (iii) are given by the following system of equations:

$$\Omega_1(g_1+g_2)\frac{\partial\Psi_1}{\partial\Omega_1} - \Omega_2(g_1-g_3)\frac{\partial\Psi_1}{\partial\Omega_2} + \Omega_3(g_3+g_4)\frac{\partial\Psi_1}{\partial\Omega_3} - \alpha_3\frac{\partial\Psi_1}{\partial z} -$$
$$\frac{\lambda_1}{\alpha_3}g_1 + ig_1'(z) - \delta\exp\left(-i\frac{\lambda_1}{\alpha_3}z\right) = 0, \quad \text{(A-10.1)}$$

$$\Omega_1(g_1+g_2)\frac{\partial\Psi_1^*}{\partial\Omega_1} - \Omega_2(g_1-g_3)\frac{\partial\Psi_1^*}{\partial\Omega_2} + \Omega_3(g_3+g_4)\frac{\partial\Psi_1^*}{\partial\Omega_3} - \alpha_3\frac{\partial\Psi_1^*}{\partial z} +$$
$$\frac{\lambda_1}{\alpha_3}g_2 - ig_2'(z) - \delta\exp\left(-i\frac{\lambda_1}{\alpha_3}z\right) = 0, \quad \text{(A-10.2)}$$

$$\Omega_1(g_1+g_2)\frac{\partial\Psi_2}{\partial\Omega_1} - \Omega_2(g_1-g_3)\frac{\partial\Psi_2}{\partial\Omega_2} + \Omega_3(g_3+g_4)\frac{\partial\Psi_2}{\partial\Omega_3} - \alpha_3\frac{\partial\Psi_2}{\partial z} -$$
$$\frac{\lambda_1}{\alpha_3}g_3 + ig_3'(z) + \delta\exp\left(-i\frac{\lambda_1}{\alpha_3}z\right) = 0, \quad \text{(A-10.3)}$$

$$\Omega_1(g_1+g_2)\frac{\partial\Psi_2^*}{\partial\Omega_1} - \Omega_2(g_1-g_3)\frac{\partial\Psi_2^*}{\partial\Omega_2} + \Omega_3(g_3+g_4)\frac{\partial\Psi_2^*}{\partial\Omega_3} - \alpha_3\frac{\partial\Psi_2^*}{\partial z} +$$
$$\frac{\lambda_1}{\alpha_3}g_4 - ig_4'(z) + \delta\exp\left(-i\frac{\lambda_1}{\alpha_3}z\right) = 0, \quad \text{(A-10.4)}$$

where $\Omega_1 \equiv (uu^*)^{-1}$, $\Omega_2 \equiv uv^{-1}$, and $\Omega_3 \equiv (vv^*)^{-1}$. The conditions on the complex-valued functions $\Psi_1$ and $\Psi_2$ in Proposition 2 (iv) and (vii) are given by the following system of equations:

$$\Omega_1(g_1+g_2)\frac{\partial\Psi_1}{\partial\Omega_1} - \Omega_2(g_1-g_3)\frac{\partial\Psi_1}{\partial\Omega_2} + \Omega_3(g_3+g_4)\frac{\partial\Psi_1}{\partial\Omega_3} - \alpha_3\frac{\partial\Psi_1}{\partial z} +$$
$$ig_1'(z) - \delta\alpha_0 = 0, \quad \text{(A-11.1)}$$

$$\Omega_1(g_1+g_2)\frac{\partial\Psi_1^*}{\partial\Omega_1} - \Omega_2(g_1-g_3)\frac{\partial\Psi_1^*}{\partial\Omega_2} + \Omega_3(g_3+g_4)\frac{\partial\Psi_1^*}{\partial\Omega_3} - \alpha_3\frac{\partial\Psi_1^*}{\partial z} -$$
$$ig_2'(z) - \delta\alpha_0 = 0, \quad \text{(A-11.2)}$$

$$\Omega_1(g_1+g_2)\frac{\partial\Psi_2}{\partial\Omega_1} - \Omega_2(g_1-g_3)\frac{\partial\Psi_2}{\partial\Omega_2} + \Omega_3(g_3+g_4)\frac{\partial\Psi_2}{\partial\Omega_3} - \alpha_3\frac{\partial\Psi_2}{\partial z} +$$
$$ig_3'(z) + \delta\alpha_0 = 0, \quad \text{(A-11.3)}$$

$$\Omega_1(g_1+g_2)\frac{\partial\Psi_2^*}{\partial\Omega_1} - \Omega_2(g_1-g_3)\frac{\partial\Psi_2^*}{\partial\Omega_2} + \Omega_3(g_3+g_4)\frac{\partial\Psi_2^*}{\partial\Omega_3} - \alpha_3\frac{\partial\Psi_2^*}{\partial z} -$$
$$ig_4'(z) + \delta\alpha_0 = 0, \quad \text{(A-11.4)}$$

where $\Omega_1 \equiv (uu^*)^{-1}$, $\Omega_2 \equiv uv^{-1}$, and $\Omega_3 \equiv (vv^*)^{-1}$. The conditions on the complex-valued functions $\Psi_1$ and $\Psi_2$ in Proposition 2 (vi) are given by the following system:

$$\Omega_1(g_1+g_2)\frac{\partial\Psi_1}{\partial\Omega_1} - \Omega_2(g_1-g_3)\frac{\partial\Psi_1}{\partial\Omega_2} + \Omega_3(g_3+g_4)\frac{\partial\Psi_1}{\partial\Omega_3} + ig_1'(z) - \delta f_0'(z) - i\frac{f_0''(z)}{f_0'(z)}g_1 = 0, \tag{A-12.1}$$

$$\Omega_1(g_1+g_2)\frac{\partial\Psi_1^*}{\partial\Omega_1} - \Omega_2(g_1-g_3)\frac{\partial\Psi_1^*}{\partial\Omega_2} + \Omega_3(g_3+g_4)\frac{\partial\Psi_1^*}{\partial\Omega_3} - ig_2'(z) - \delta f_0'(z) + i\frac{f_0''(z)}{f_0'(z)}g_2 = 0, \tag{A-12.2}$$

$$\Omega_1(g_1+g_2)\frac{\partial\Psi_2}{\partial\Omega_1} - \Omega_2(g_1-g_3)\frac{\partial\Psi_2}{\partial\Omega_2} + \Omega_3(g_3+g_4)\frac{\partial\Psi_2}{\partial\Omega_3} + ig_3'(z) + \delta f_0'(z) - i\frac{f_0''(z)}{f_0'(z)}g_3 = 0, \tag{A-12.3}$$

$$\Omega_1(g_1+g_2)\frac{\partial\Psi_2^*}{\partial\Omega_1} - \Omega_2(g_1-g_3)\frac{\partial\Psi_2^*}{\partial\Omega_2} + \Omega_3(g_3+g_4)\frac{\partial\Psi_2^*}{\partial\Omega_3} - ig_4'(z) + \delta f_0'(z) + i\frac{f_0''(z)}{f_0'(z)}g_4 = 0, \tag{A-12.4}$$

where $\Omega_1 \equiv (uu^*)^{-1}$, $\Omega_2 \equiv uv^{-1}$, and $\Omega_3 \equiv (vv^*)^{-1}$.

## References


[1] Ovsiannikov L.V., Group Analysis of Differential Equations, Ed. Ames W.F., Academic Press, New York, 1982.

[2] Olver P.J., Applications of Lie Groups to Differential Equations, Springer, New York, 1986.

[3] Euler N. and Steeb W.-H., Continuous Symmetries, Lie Algebras and Differential Equations. Wissenschaftsverlag, Mannheim, 1992.

[4] Fushchych W.I., Shtelen W.M. and Serov N.I., Symmetry Analysis and Exact Solutions of Equations of Nonlinear Mathematical Physics. Kluwer Academic Publishers, Dordrecht, 1993.

[5] Menyuk C.R., Stability of Solitons in Birefringent Optical Fibres. I: Equal Propagation Amplitudes, *Optical Lett.*, 1987, V.12, 614–616.

[6] Menyuk C.R., Nonlinear Pulse Propagation in Birefringent Optical Fibres, *IEEE J. Quantum Electronics*, 1987, V.23, 174–176.

[7] Menyuk C.R., Stability of Solitons in Birefringent Optical Fibres. II: Arbitrary Amplitudes, *J. Opt. Soc. Am. B*, 1988, V.5, 392–402.

[8] Menyuk C.R., Pulse Propagation in an Elliptically Birefringent Kerr Medium, *IEEE J. Quantum Electronics*, 1989, V.25, 2674–2682.

[9] Malomed B., Polarization Dynamics and Interactions of Solitons in a Birefringent Optical Fibre, *Physical Review A*, 1991, V.43, 410–423.

[10] Malomed B., Bound Solitons in Coupled Nonlinear Schrödinger Equations, *Physical Review A*, 1992, V.45, R8321–R8323.

[11] Mahmood M.F., Gill T.L. and Zachary W.W., An Analytic Approach to Nonlinear Pulse Propagation in Birefringent Optical Fibres, Submitted to *Phys. Rev. A*, 1994.

[12] Fushchych W. and Cherniha R., Systems of Nonlinear Evolution Equations of the second order invariant under the Galilean Algebra, *Dopovidi Ukrainian Acad. Sci.*, 1993, N 8, 44.