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q-Deformed Dressing Operators And Modified Integrable Hierarchies

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Abstract

A q-deformation of the dressing operator introduced by Sato is suggested. It is shown that it produces q-deformation of known integrable heirarchies, with the infinite number of conservation laws. A modification introduced by Kupershmidt when incorporated leads to both modified and deformed integrable systems.

One of the most important goal of research in a nonlinear integrable system is to extend the class of equations of such a type. Recently a new class of integrable equations has emerged out of the attempt to quantize the existing integrable hierarchies [1]. These are called q-deformed nonlinear systems which retain many of beautiful properties of usually integrable systems. An important reason for studying these q-deformed systems is the recent occurrence of q-deformed Lie algebra in the literature [2].

We start by defining the q-deformed differential operator $\tilde{D}$ as

$$\tilde{D} = \frac{(1-Q^{-2})}{(1-q^{-2})z}, \tag{1}$$

where q is the deformation parameter and Q is a formal differential operator of an infinite order in the ordinary co-ordinate variable:

$$Q = \sum_{n=0}^{\alpha} \frac{1}{n!}(-1)^n \varepsilon^n z^n \partial^n, \tag{2}$$

where $\varepsilon = 1 - q$, Q has the action

$$Qf(z) = f(zq). \tag{3}$$

Actually this was the definition originally followed by Exton [3] and Jackson [4]. In the q-deformed Sato's [5] approach we start with the pseudo-differential operator

$$L = \tilde{D} + a_0 + a_1 \tilde{D}^{-1} + a_2 \tilde{D}^{-2} + \dots \tag{4}$$

and also the underessed one (or bare)

$$L_0 = \tilde{D}. \tag{5}$$

As in the case of usual Sato's approach we begin with a dressing operator S

$$S = 1 + S_1 \tilde{D}^{-1} + S_2 \tilde{D}^{-2} + S_3 \tilde{D}^{-3} + \cdots \tag{6}$$

and demand that

$$L = S L_0 S^{-1}. \tag{7}$$

Using the Leibniz rule for deformed operators we can at once obtain from equation (7)

$$\begin{array}{rcl} a_0 &=& (1 - Q^{-2}) S_1, \\ a_1 &=& (1 - Q^{-2}) S_2 - a_0 S_1 - \tilde{D} S_1, \\ a_2 &=& (1 - Q^{-2}) S_3 - (\tilde{D} S_2) - a_0 S_2 - a_1 S_1^{(0.2)} \end{array} \tag{8}$$

etc., where for any function $f(z)$ we denote by $f^{(n,m)}$ the following

$$f^{(n,m)} = (\tilde{D}^n Q^m f(z)) \tag{9}$$

in the limit of $q \to 1$, when $\tilde{D} \to \partial, Q \to 1$, we get usual Sato's relations;

$$\begin{array}{rcl} a_0 &=& 0, \\ a_1 &=& -S_1 x, \\ a_2 &=& -S_{2x} + S_1 S_{1x} \quad \text{etc.} \end{array} \tag{10}$$

The time evolution operator is obtained by dressing up any power of $\tilde{D}$, that is $\tilde{D}^m$,

$$M = S \tilde{D}^m S^{-1} \tag{11}$$

along with the obvious condition

$$S S^{-1} = 1. \tag{12}$$

If we write

$$S^{-1} = 1 + \alpha_1 \tilde{D}^{-1} + \alpha_2 \tilde{D}^2 + \alpha_3 \tilde{D}^{-3} + \cdots, \tag{13}$$

then equation (12) leads to the following relations:

$$\begin{array}{rcl} \alpha_1 &=& -S_1, \\ \alpha_2 &=& -S_2 + S_1 S_1^{(0,2)}, \\ \alpha_3 &=& -S_3 - S_2 \alpha_1^{(0,4)} - S_1 \alpha_2^{(0,2)} + q^{-2} S_1 \alpha_1^{(0,4)} \end{array} \tag{14}$$

and so on, thus determining uniquely the inverse operator.

As a particular example we take the third order flow and consider

$$\begin{array}{rcl} M &=& S \tilde{D}^3 S^{-1} \\ &=& \tilde{D}^3 + [\alpha_1^{(0,-6)} + S_1] \tilde{D}^2 + P \tilde{D} + Q, \end{array} \tag{15}$$

where

$$\begin{aligned} P &= (q^4+q^2+1)\alpha_1^{(1,-4)}+\alpha_2^{(0,-6)}+S_1\alpha_1^{(0,-4)}+S_2, \\ Q &= (q^4+q^2+1)\Big(\alpha_1^{(2,-2)}+\alpha_2^{(1,-4)}\Big)+\alpha_3^{(1,-6)}+ \\ & \alpha_4 S_1\{(q^2+1)\alpha_1^{(1,-2)}+\alpha_2^{(0,-4)})\}+S_2\alpha_1^{(0,-2)}+S_3\ldots \end{aligned} \tag{16}$$

At this point it will be interesting to see the connection with the q-deformed Gelfand-Dikki prescription for generation of integrable flows. There one starts from a pseudodifferential operator

$$K^{1/2}=\tilde{D}+\sum_{n=0}^{\alpha}W_{-n}\tilde{D}^{-n}, \tag{17}$$

where $\tilde{D}.\tilde{D}^{-1}=\tilde{D}^{-1}\tilde{D}=1$. The identification of the coefficients W_{-n} are obtained by constructing integer powers of $K^{1/2}$. For example

$$(K^{1/2}K^{1/2})_+=K_+=\tilde{D}^2+V_1\tilde{D}+V_0, \tag{18}$$

where ' + ' denotes only the positive part of the product. The coefficients W_{-i} are then determined as

$$\begin{aligned} W_0 &= (1+Q^{-2})^{-1}V_1, \\ W_{-1} &= -(1+Q^{-2})^{1}(-V_0+W_0^{(1,0)}+W_0^2), \\ W_{-2} &= -(1+Q^{-2})^{-1}(W_{-1}W_0^{(0,2)}+W_{-1}^{(1,0)}+W_0W_{-1}), \end{aligned} \tag{19}$$

and so on. One can proceed in a similar manner and construct $K^{3/2}$ to obtain

$$\begin{aligned} K^{3/2} &= \tilde{D}^3+(V_1+W_0^{(0,-4)})\tilde{D}^2+[(q^2+1)W_0^{(1,-2)}+W_{-1}^{(0,-4)}+ \\ & V_1W_0^{(0,-2)}+V_0]\tilde{D}+[W_0^{(2,0)}+(q^2+1)W_{-1}^{(1,-2)}+ \\ & W_{-2}^{(0,-4)}+V_1W_0^{(1,0)}+V_1W_{-1}^{(0,-2)}+V_0W_0]+(\ldots)\tilde{D}^{-1}. \end{aligned} \tag{20}$$

Note that in equation (15) we also constructed the third order operator by the dressing approach. We can check that the two methods lead to the same equations, for example, in (15)

$$\begin{aligned} \alpha_1^{(0,-6)}+S_1 &= (1-Q^{-6})S_1=(1-Q^{-6})(1-Q^{-2})^{-1}a_0 \\ &= (1+Q^{-4}+Q^{-2})a_0. \end{aligned} \tag{21}$$

On the other hand, from equation (20) the coefficient of $\tilde{D}^2$ is

$$V_1 \quad +W_0^{(0,-4)}=(1+Q^{-2})W_0+Q^{-4}W_0=(1+Q^{-4}+Q^{-2})W_0. \tag{22}$$

So if, W_0 is identified with a_0, these are same. Similar verifications can also be done for other coefficients. Now we turn to the question of the construction of deformed integrable systems and to their modified forms. It is well known that integrable systems can be constructed from the equations

$$\frac{dK}{dt} = \Big[\underset{\geq 0}{K^{n/2}}, \quad K \Big], \tag{23}$$

and their modified form via

$$\frac{dK}{dt} = \Big[\underset{\geq 1}{K^{3/2}} \quad K \Big]. \tag{24}$$

The computation of the comutator proceeds with the help of formulae

$$\tilde{D}^n o f(z) = \sum_{m=0}^{n} \Big[\binom{n}{m} \Big] q^{2m(n-m)} f^{(m,2m-2n)} \tilde{D}^{n-m} \tag{25}$$

where

$$\begin{aligned} \Big[\binom{n}{m} \Big] &= \frac{[n]!}{[m]![n-m]!}, \\ [m]! &= [m][m-1]\ldots\ldots[2][1], \\ [m] &= \frac{1-q^{-2m}}{1-q^{-2}}. \end{aligned}$$

If we now substitute (20) in equation (24) we immediately obtain

$$\begin{aligned} \frac{\partial V_0}{\partial t} = {} & V_0^{(3,0)} + V_1 V_2^{(2,0)} + W_0^{(0,-4)} V_0^{(2,0)} + V_0 V_0^{(1,0)} + W_{-1}^{(0,-4)} V_0^{(1,0)} + (q^2+1) \times \\ & W_0^{(1,-2)} V_0^{(1,0)} + V_1 W_0^{(0,-2)} V_0^{(1,0)} \end{aligned} \tag{26}$$

along with

$$\begin{aligned} \frac{\partial V_1}{\partial t} = {} & V_1^{(3,0)} + (q^4+q^2+1) V_0^{(2,-2)} + V_1 V_1^{(2,0)} + W_0^{(0,-4)} V_1^{(2,0)} + \\ & (q^2+1) V_1 V_0^{(1,-2)} + (q^2+1) W_0^{(0,-4)} V_0^{(1,-2)} - V_0^{(2,0)} + \\ & V_0 V_1^{(1,0)} - V_1 V_0^{(1,0)} + (V_0 V_0^{(0,-2)} - V_0^2) - W_{-1}^{(2,-4)} + \\ & W_{-1}^{(0,-4)} V_1^{(1,0)} - V_1 W_{-1}^{(1,-4)} + W_{-1}^{(0,-4)} V_0^{(0,-2)} - \\ & V_0 W_{-1}^{(0,-4)} - (q^2+1) W_0^{(3,-2)} + (q^2+1)[W_0^{(1,-2)} V_1^{(1,0)} - \\ & V_1 W_0^{(2,-2)}] + (q^2+1)[W_0^{(1,-2)} V_0^{(0,-2)} - V_0 W_0^{(1,-2)}] - \\ & V_1^{(2,0)} W_0^{(0,-2)} - (q^2+1) V_1^{1,-2)} W_0^{(1,-2)} - V_1^{(0,-4)} W_0^{(2,-2)} - \\ & V_1^{(0,-2)} W_0^{(1,-2)} V_1 + V_1 W_0^{(0,-2)} V_0^{(0,-2)} - V_0 V_1 W_0^{(0,-2)} \ldots \end{aligned} \tag{27}$$

If we consider the limit $q \to 1$, $\varepsilon \to 0$ then

$$\begin{aligned} W_0 &= 1/2V_1, \\ W_{-1} &= 1/2(V_0 - 1/2V_{1,x} - 1/4V_1^2), \\ W_{-2} &= -1/2(W_{-1x} + 2W_0W_{-1}). \end{aligned} \tag{28}$$

Then equations (26) and (27) go over to the following two coupled equations first obtained by Konopelchenko and Oevel [6]

$$\begin{aligned} 8V_{0t} &= 8V_{0xxx} + 12V_1V_{0xx} + 12V_0V_{0x} + 6V_{0x}V_{1x} + 3V_1^2V_{0x}, \\ 8V_{1t} &= 2V_{1xxx} + 12V_{0xx} + 12(V_1V_{0x} + V_0V_{1x}) - 3V_1^2V_{1x}. \end{aligned} \tag{29}$$

One of the most important properties of integrable systems is that they possess the infinite number of conservation laws. It can be demonstrated that the q-deformed equations have the same property. For any pseudo-differential operator K we denote by

$$K_+ = \sum_{n=0}^{M} k_n\tilde{D}^n, \quad K_- = \sum_{n=1}^{\infty} k_{-n}\tilde{D}^{-n} \tag{30}$$

and res $K = K_{-1}$. According to the theorem by Drinfeld and Sokolov, if P and Q are formal pseudo-differential operators then res $[P, Q]$ is a total derivative of some differential polynomial in the coefficients of P and Q. We can extend this theorem to the case of q-deformed differential operators, because they can be expanded as series ordinary differential operators of positive and negative orders. Let

$$P = \sum_{-\infty}^{\infty} \tilde{a}_m\partial^m, \qquad Q = \sum_{-\infty}^{\infty} \tilde{b}_l\partial^l. \tag{31}$$

Then Res $[P, Q] = \dfrac{\partial q}{\partial z}$, where

$$\begin{aligned} q = \sum_{m,1=-\infty}^{\infty} \frac{m(m-1)\ldots\ldots(2-1)(i-1)(-1)}{(m+1+1)!} &\left\{\sum_{i=0}^{m+1}(-1)^i\tilde{a}_m^{(i)}\tilde{b}_1^{m+1-i}\right\}, \\ \ldots& \\ \tilde{a}^i = (\partial^i\tilde{a}).& \end{aligned} \tag{32}$$

Now for any Lax equation

$$\frac{dL}{dt} = [A, L], \tag{33}$$

where $L = \tilde{D}^N + \sum_{n=0}^{W-1} V_n\tilde{D}^n$, we get $\dfrac{dL^{r/k}}{dt} = [A, L^{r/k}]$ and

$$\frac{d}{dt}\text{ res } L^{r/k} = \text{res } [A, L^{r/k}] = \frac{\partial f}{\partial z}(\text{say}). \tag{34}$$

Integrating, we obtain

$$\frac{d}{dt}\int \text{res } L^{r/k} dz = 0, \tag{35}$$

so conserved quantities are

$$C_r = \int dz(\text{res} L^{r/k}). \tag{36}$$

In our case considered above we have

$$\text{res } (K^{3/2}) = W_{-1}^{(2,0)} + (q^2+1)W_{-2}^{(1,-2)} + V_0 W_{-1} + W_{-3}^{(0,-4)} + V_1[W_{-1}^{(1,0)} + W_{-2}^{(0,-2)}]$$

which reproduces in the limit $q \to 1$ the previously known conserved quantities.

In our above discussions we have shown that both the dressing operator and fractional power approach can be used for the q-deformed integrable systems. The dressing operator method may be used to generate actual solutions of the q-deformed system.

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