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Coisotropic quasi-periodic motions near the relative equilibrium of a Hamiltonian system

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Abstract

We consider the Hamiltonian system which is invariant under locally Hamiltonian (non-Poissonian) action of torus. We show that when a certain set of conditions is satisfied the majority of motions in a sufficiently small neighbourhood of system's relative equilibrium are quasi-periodic and cover coisotropic invariant tori.

Let (M, ω^2) be a smooth $2n$-dimensional symplectic manifold admitting free smooth symplectic action of k-dimensional torus T^k. The projection

$$\pi \ : \ \mathsf{M} \ \mapsto \ \mathsf{N} \sim \mathsf{M}/\mathsf{T}^k$$

determines the structure of a principal T^k-fibre bundle $(\mathsf{M}, \mathsf{N}, \pi)$. Consider a Hamiltonian system on M with the T^k-invariant Hamiltonian function $H = \tilde{H} \circ \pi$, $\tilde{H} \ : \ \mathsf{N} \mapsto \mathbf{R}$. Our goal is to investigate motions of such a system in neighbourhoods of its relative equilibria in the case where T^k-action does not admit the momentum map. Recall that the relative equilibrium (r.e.) is such a trajectory of motion that coincides with an action orbit of the one-parameter subgroup of system's symmetry group [1–3]. In the case where the momentum map exists the r.e. is usually found applying symplectic reduction, taking into account that the projection of r.e. is exactly the equilibrium of the reduced system [1–4].

In this paper we consider the case where the 2-cocycle of T^k-action on M is nontrivial, thus even on the universal covering $\hat{\mathsf{M}}$ of M the induced momentum map (which always exists on $\hat{\mathsf{M}}$) is not Ad^*-equivariant. To overcome this difficulty we use the reduction procedure developed in [5–8].

In section 1 we show that, when a certain set of conditions is satisfied, the Hamiltonian system in a neighbourhood of r.e. may be regarded as close to an integrable one. The systems of a mechanical type with gyroscopic forces are discussed in more detail in section 2. In section 3 the results of [9] are applied to establish the KAM-like theorem on the existence of coisotropic quasi-periodic motions in a neighbourhood of r.e., and in section 4 we investigate the system of three constrained axisymmetrical rotors with the gyroscopic interaction.

1 Asymptotic integrability of the T^k-invariant Hamiltonian system near the relative equilibrium

Denote by t^k the Lie algebra of torus T^k, and by X_a the vector field on M which generates action of the torus one-parameter subgroup corresponding to $\boldsymbol{a} \in \mathsf{t}^k$. The 2-cocycle of T^k-action is determined by

$$\mathcal{C}(\boldsymbol{a}, \boldsymbol{b}) = \omega^2(X_a, X_b).$$

The projection $\pi : \mathsf{M} \mapsto \mathsf{N}$ gives rise to the reduced Poisson structure (r.P.s.) on N. In particular, if $H = \tilde{H} \circ \pi$ is the T^k-invariant Hamiltonian function, then the projection of the Hamiltonian vector field $\Im dH$ on M is the Hamiltonian (with respect to r.P.s) vector field $\Im_N d\tilde{H}$ on N. There exists a K^*-valued closed 1-form $\boldsymbol{\theta}$ on M ($K = \ker \mathcal{C}$) such that

$$(\boldsymbol{\theta}(\cdot)|\boldsymbol{a}) = -\omega^2(X_a, \cdot) \quad \forall \boldsymbol{a} \in K.$$

This 1-form drops on N, i.e. there exists the 1-form $\tilde{\boldsymbol{\theta}}$ on N satisfying $\pi^* \tilde{\boldsymbol{\theta}} = \boldsymbol{\theta}$. The symplectic leaves of r.P.s are determined by the Pfaff equation $\tilde{\boldsymbol{\theta}} = 0$ (see [5–7] for more information). Let $\ell(x_0)$ be the symplectic leaf passing through $x_0 \in \mathsf{N}$. In a neighbourhood $\mathcal{U}(x_0)$ of x_0 one can define the map $\tilde{\boldsymbol{J}} : \mathcal{U}(x_0) \mapsto K^*$, which is a local potential of the 1-form $\boldsymbol{\theta}$, and $\tilde{\boldsymbol{J}}(x_0) = 0$. Thus $\ell(x_0)$ locally coincides with $\tilde{\boldsymbol{J}}^{-1}(0)$. The components of $\tilde{\boldsymbol{J}}$ with respect to some basis of the space K^* are the local Casimir functions for the r.P.s.

Let x_0 be a critical point of $\tilde{H}|_{\ell(x_0)}$. Then x_0 is the equilibrium for restriction of the Hamiltonian system $\dot{x} = \Im_N d\tilde{H}$ to $\ell(x_0)$. Suppose that this equilibrium is stable in linear approach. Then eigenvalues of the linear operator $(\Im_N d\tilde{H})_*|_{T_{x_0}\ell(x_0)}$ are of the form

$$\pm \mathrm{i}\lambda_1^0, ..., \pm \mathrm{i}\lambda_m^0, \quad m := \dim \ell(x_0)/2 = (\dim \mathsf{N} - k_0)/2,$$

λ_j^0 being real. It is well known that there exists a symplectic basis of the space $T_{x_0}\ell(x_0)$ in which the matrix of a quadratic form $d^2\tilde{H}|_{T_{x_0}\ell(x_0)}$ becomes

$$\mathrm{diag}(\lambda_1^0, ..., \lambda_m^0).$$

In this case the point x_0 will be called stable and $\lambda_1^0, ..., \lambda_m^0$ will be called *eigenfrequencies of the Hamiltonian* $\tilde{H}$ at the point x_0.

Theorem 1 *Let x_0 be the stable critical point of $\tilde{H}|_{\ell(x_0)}$ and the eigenfrequencies of Hamiltonian $\tilde{H}$ at x_0 satisfy the nonresonant condition up to the order $2l$ inclusive, i.e.*

$$k_1\lambda_1^0 + \cdots + k_m\lambda_m^0 \neq 0$$

for all $k_j \in \mathbf{Z}$, $j = 1, ..., m$, $0 < \sum_{j=1}^m |k_j| \le 2l$.

Then there exists the neighbourhood $\mathcal{V}(x_0) \in \mathsf{N}$ of x_0 such that:

1. *The portion* $\pi^{-1}(\mathcal{V}(x_0))$ *is diffeomorphic to the direct product*

$$\mathcal{B}^m(0;R_I)\times\mathcal{B}^{k_0}(0;R_J)\times\mathsf{T}^m\times\mathsf{T}^k,$$

where R_I, R_J *are some positive numbers,*

$$\mathcal{B}^m(0;R_I)=\{\boldsymbol{I}=(I_1,...,I_m)\in\mathbf{R}^m:\ \|\boldsymbol{I}\|<R_I\},$$

$$\mathcal{B}^{k_0}(0;R_I)=\{\boldsymbol{J}=(J_1,...,J_{k_0})\in\mathbf{R}^{k_0}:\ \|\boldsymbol{J}\|<R_J\},$$

$$\mathsf{T}^m=\mathsf{T}^m_\varphi=\{\boldsymbol{\varphi}=(\varphi_1,...,\varphi_m)|\mathrm{mod}\,2\pi\},$$

$$\mathsf{T}^k=\mathsf{T}^k_\psi=\{\boldsymbol{\psi}=(\psi_1,...,\psi_k)|\mathrm{mod}\,2\pi\}.$$

2. *The Hamiltonian* H *in coordinates* $(\boldsymbol{I},\boldsymbol{J},\boldsymbol{\varphi},\boldsymbol{\psi})$ *takes the form*

$$\begin{aligned} H \quad = \quad & H^0(\boldsymbol{J})+\sum_{j=1}^m\lambda_j(\boldsymbol{J})I_j+\sum_{2\le j_1+\dots+j_m\le l}\lambda_{j_1,...,j_m}(\boldsymbol{J})I_1^{j_1}\cdots I_m^{j_m}+\\ & +\mathcal{O}(\|\boldsymbol{I}\|^{l+1}), \end{aligned} \tag{1}$$

where $H^0(\boldsymbol{J}),\lambda_j(\boldsymbol{J}),\lambda_{j_1,...,j_m}(\boldsymbol{J})$ *are some smooth functions defined in* $\mathcal{B}^{k_0}(0;R_J)$, $\lambda_j(\boldsymbol{J})$ *satisfying the condition* $\lambda_j(0)=\lambda_j^0,\ j=1,...,m.$

3. *The matrix of Poisson brackets for the above coordinates does not depend on* $\boldsymbol{\varphi}$ *and* $\boldsymbol{\psi}$*, in particular,*

$$\{\boldsymbol{I},\boldsymbol{J}\}=0;\quad \{\varphi_i,I_j\}=\delta_{ij};\quad \{\boldsymbol{\varphi},\boldsymbol{J}\}=0;$$

$$\{\boldsymbol{\psi},I_i\}=\sum_{j=1}^{k_0}\frac{\partial V_i(\boldsymbol{J})}{\partial J_j}\boldsymbol{\sigma}_j;\quad \{\boldsymbol{\psi},J_j\}=\boldsymbol{\sigma}_j, \tag{2}$$

where $\{\boldsymbol{\sigma}_j=(\sigma_{i,j},...,\sigma_{k,j})\}_{j=1}^{k_0}$ *is some basis of* $K\subset\mathsf{t}^k$, $V_i(\boldsymbol{J})$ *are some smooth functions defined on* $\mathcal{B}^{k_0}(0;R_J)$.

P r o o f. Consider the neighbourhood $\mathcal{U}(x_0)$ of x_0 for which $\pi^{-1}(\mathcal{U}(x_0))\sim\mathcal{U}(x_0)\times\mathsf{T}^k_\psi$. From now on we will not distinguish between functions $f:\mathcal{U}(x_0)\mapsto\mathbf{R}$ and $f\circ\pi$. Let us introduce coordinates

$$(\boldsymbol{u},\boldsymbol{J})=(u_1,...,u_{2m},J_1,...,J_{k_0})$$

in $\mathcal{U}(x_0)$, where J_j denotes the j-th component of the map $\boldsymbol{J}$ in the basis of K^* dual to $\{\boldsymbol{\sigma}_j\}$. Obviously, $\{\boldsymbol{\psi},J_j\}=\boldsymbol{\sigma}_j$.

By the implicit function theorem, the equation

$$\frac{\partial H(\boldsymbol{u},\boldsymbol{J})}{\partial\boldsymbol{u}}=0$$

determines the local manifold $\boldsymbol{u}=\boldsymbol{u}^*(\boldsymbol{J})$ of singular points for the vector field $\Im_N dH$. Introducing new variables $\boldsymbol{v}=\boldsymbol{u}-\boldsymbol{u}^*(\boldsymbol{J})$ we obtain

$$H=H^0(\boldsymbol{J})+\frac{1}{2}\left(\frac{\partial^2H(\boldsymbol{u}^*(\boldsymbol{J}),\boldsymbol{J})}{\partial\boldsymbol{u}^2}\boldsymbol{v},\boldsymbol{v}\right)+\mathcal{O}(\|\boldsymbol{v}\|^3),$$

where $H^0(\boldsymbol{J}) = H(\boldsymbol{u}^*(\boldsymbol{J}), \boldsymbol{J})$. According to the well known result of Lie (see, for example, [10]) we may take the matrix of Poisson brackets for the variables $(\boldsymbol{v}, \boldsymbol{J})$ to be of the form $\operatorname{diag}(\mathbf{I}_{2m}, \mathbf{0}_{k_0})$, where

$$\mathbf{I}_{2m} = \begin{pmatrix} \mathbf{0}_m & -\mathbf{E}_m \\ \mathbf{E}_m & \mathbf{0}_m \end{pmatrix},$$

i.e. $v_1, ..., v_{2m}$ are canonical coordinates on each symplectic leaf $\boldsymbol{J} = \text{const}$. Since $\lambda_i^0 \neq \lambda_j^0$, $i \neq j$, the matrix $\frac{\partial^2 H(u^*(J),J)}{\partial u^2}$ can be reduced to the diagonal form by means of a linear canonical transformation which is smoothly dependent on $\boldsymbol{J}$ [11]. Thus there exist coordinates

$$(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{J}) = (p_1, ..., p_m, q_1, ..., q_m, J_1, ..., J_{k_o}),$$

such that

$$H = H^0(\boldsymbol{J}) + \frac{1}{2}\sum_{j=1}^{m} \lambda_j(\boldsymbol{J})(p_j^2 + q_j^2) + \mathcal{O}(\|(\boldsymbol{p}, \boldsymbol{q})\|^3),$$

where $\lambda_j(\boldsymbol{J})$ are some smooth functions defined in the vicinity of origin of the $\boldsymbol{J}$-space, $\pm \mathrm{i}\lambda_j(\boldsymbol{J})$ being eigenvalues of the matrix $\mathbf{I}_{2m}\frac{\partial^2 H(u^*(J),J)}{\partial u^2}$. The symplectic structure on the leaf $\boldsymbol{J} = \text{const}$ takes the standard form $d\boldsymbol{p} \wedge d\boldsymbol{q}$.

Now introduce canonical coordinates $(\boldsymbol{I}, \boldsymbol{\varphi})$ by

$$p_j = \sqrt{2I_j}\cos\varphi_j, \quad q_j = \sqrt{2I_j}\sin\varphi_j.$$

Observe that $\{\varphi_i, I_j\} = \delta_{i,j}$ and all other pairwise combinations of the Poisson brackets vanish.

We choose R_J so that the nonresonant properties

$$k_1\lambda_1(\boldsymbol{J}) + \cdots + k_m\lambda_m(\boldsymbol{J}) \neq 0$$

hold true in $\mathcal{B}^{k_0}(0; R_J)$ for all $k_j \in \mathbf{Z}$, $j = 1, ..., m$, $0 < \sum_{j=1}^m |k_j| \leq 2l$. Then, without loss of generality, we may take the Hamiltonian H to be reduced to the normal form (1) [3].

We shall find the t^k-valued function $\boldsymbol{F}(\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{\varphi})$ and scalar functions $V_i(\boldsymbol{J})$ such that (2) will hold after transformation $\boldsymbol{\psi} \mapsto \boldsymbol{\psi} + \boldsymbol{F}$. Let L be the subspace of t^k additional with respect to K, so that $\mathsf{t}^k = K \oplus L$. Then

$$\{\boldsymbol{\psi}, I_i\} = \sum_{j=1}^{k_0} \xi_{ij}(\boldsymbol{I}, \boldsymbol{J})\boldsymbol{\sigma}_j + \boldsymbol{\gamma}_j(\boldsymbol{I}, \boldsymbol{J}) + \boldsymbol{f}_i(\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{\varphi}),$$

where ξ_{ij}, $\boldsymbol{\gamma}_j$, $\boldsymbol{f}_i$ are respectively scalar, L-valued, and t^k-valued functions, $\int_{\mathrm{T}^m_\varphi} \boldsymbol{f}_i d\boldsymbol{\varphi} = 0$. From the Jacobi identity

$$\{\{\boldsymbol{\psi}, I_i\}, I_j\} + \{\{I_i, I_j\}, \boldsymbol{\psi}\} + \{\{I_j, \boldsymbol{\psi}\}, I_i\} = 0$$

it follows that

$$\frac{\partial \boldsymbol{f}_i}{\partial \varphi_j} = \frac{\partial \boldsymbol{f}_j}{\partial \varphi_i}.$$

For this reason there exists the t^k-valued function $\boldsymbol{F}(\boldsymbol{I},\boldsymbol{J},\boldsymbol{\varphi})$ such that $\boldsymbol{f}_i=\frac{\partial \boldsymbol{F}}{\partial\varphi_i}$. After the transformation $\boldsymbol{\psi}\mapsto\boldsymbol{\psi}+\boldsymbol{F}$ we obtain

$$\{\boldsymbol{\psi},I_i\}=\sum_{j=1}^{k_0}\xi_{ij}(\boldsymbol{I},\boldsymbol{J})\boldsymbol{\sigma}_j+\gamma_i(\boldsymbol{I},\boldsymbol{J}). \tag{3}$$

Let us prove that $\boldsymbol{\gamma}_j=0$. As has been shown in [6], there exists the connection form $\boldsymbol{\omega}$ of the principal T^k-fibre bundle $(\mathsf{M},\mathsf{N},\pi)$ such that

$$\omega^2=\mathrm{h}\omega^2+\boldsymbol{\theta}\wedge\mathcal{P}_K\boldsymbol{\omega}+\mathcal{C}(\boldsymbol{\omega},\boldsymbol{\omega}), \tag{4}$$

holds, where $\mathrm{h}\omega^2$ is the horizontal part of ω^2 with respect to $\boldsymbol{\omega}$, $\boldsymbol{\theta}$ is the K^*-valued closed 1-form, which has been already defined above, $\mathcal{P}_K$ is the projection on the subspace K along L (The condition $\iota(X_a)\omega^2=\mathcal{C}(\boldsymbol{a},\boldsymbol{\omega})$, $\forall\boldsymbol{a}\in L$, determines the above connection form uniquely up to the component $\mathcal{P}_K\boldsymbol{\omega}$). Observe that $\boldsymbol{\theta}(X_a)=0$ holds for all $\boldsymbol{a}\in\mathsf{t}^k$. Let X_i be the vector field with the Hamiltonian I_i. In the $\boldsymbol{\psi}$-coordinates the vertical component of this vector field (with respect to $\boldsymbol{\omega}$) is given by (3). Denoting by $\boldsymbol{h}X_i$ the horizontal component of X_i, we obtain from (4)

$$\omega^2(X_i,X_a)=\boldsymbol{\theta}(\boldsymbol{h}X_i)\mathcal{P}_K\boldsymbol{a}+\mathcal{C}(\boldsymbol{\gamma}_i,\boldsymbol{a})=\mathcal{C}(\boldsymbol{\gamma}_i,\boldsymbol{a}),\quad\forall\boldsymbol{a}\in L.$$

On the other hand

$$\omega^2(X_i,X_a)=-dI_i(X_a)=0.$$

For this reason

$$\boldsymbol{\gamma}_i=0. \tag{5}$$

Now let us show that ξ_{ij} does not depend on $\boldsymbol{I}$. Observe that the invariance of the Poisson brackets with respect to shifts of the $\boldsymbol{\psi}$-coordinates implies the function $\{\varphi_l,\boldsymbol{\psi}\}$ to be independent on $\boldsymbol{\psi}$, $l=1,...,k$. So, taking into account (3), (5), and the Jacobi identity

$$\{\{\boldsymbol{\psi},I_i\},\varphi_l\}+\{\{I_i,\varphi_l\},\boldsymbol{\psi}\}+\{\{\varphi_l,\boldsymbol{\psi}\},I_i\}=0,$$

we obtain

$$\{\sum_{j=1}^{k_0}\xi_{ij}\boldsymbol{\sigma}_j,\varphi_l\}+\frac{\partial}{\partial\varphi_i}\{\varphi_l,\boldsymbol{\psi}\}=0.$$

From this it follows the required property $\frac{\partial\xi_{ij}}{\partial I_l}=0$ and, besides that,

$$\frac{\partial}{\partial\varphi_i}\{\varphi_l,\boldsymbol{\psi}\}=0,\quad l=1,...,m,\ i=1,...,m.$$

Now let us show that the equalities

$$\frac{\partial\xi_{ij}}{\partial J_l}=\frac{\partial\xi_{il}}{\partial J_j}. \tag{6}$$

hold true. Using the identity

$$\{\{\boldsymbol{\psi},I_i\},\psi_n\}+\{\{I_i,\psi_n\},\boldsymbol{\psi}\}+\{\{\psi_n,\boldsymbol{\psi}\},I_i\}=0,$$

and taking into account (3), (5), we obtain

$$\sum_{j=1}^{k_0}\left(\sum_{l=1}^{k_0}\frac{\partial \xi_{ij}}{\partial J_l}\{J_l,\psi_n\}\right)\boldsymbol{\sigma}_j+\{\boldsymbol{\psi},\sum_{j=1}^{k_0}\xi_{ij}(\boldsymbol{J})\sigma_{n,j}\}+\frac{\partial}{\partial\varphi_i}\{\psi_n,\boldsymbol{\psi}\}=0. \tag{7}$$

From this it follows

$$-\sum_{j=1}^{k_0}\left(\sum_{l=1}^{k_0}\frac{\partial \xi_{ij}}{\partial J_l}\sigma_{n,l}\right)\boldsymbol{\sigma}_j+\sum_{l=1}^{k_0}\left(\sum_{j=1}^{k_0}\frac{\partial \xi_{ij}}{\partial J_l}\sigma_{n,j}\right)\boldsymbol{\sigma}_l=0.$$

That is

$$\sum_{l=1}^{k_0}\left(\sum_{j=1}^{k_0}\left(\frac{\partial \xi_{ij}}{\partial J_l}-\frac{\partial \xi_{il}}{\partial J_j}\right)\sigma_{n,j}\right)\boldsymbol{\sigma}_l=0,$$

This implies (6). Thus there exist functions $V_i(\boldsymbol{J})$ in $\mathcal{B}^{k_0}(0;R_J)$ such that

$$\xi_{ij}=\frac{\partial V_i(\boldsymbol{J})}{\partial J_j},\quad i,j=1,...,m.$$

From (7) it also follows that

$$\frac{\partial}{\partial\varphi_i}\{\psi_n,\boldsymbol{\psi}\}=0,\quad i=1,...,m;\ n=1,...,k.$$

The coordinates constructed above satisfy the statement 3 of the theorem, and the Hamiltonian is of the form (1). □

Remark One can construct the functions $V_i(\boldsymbol{J})$ in such a way that $V_i(0)=0$ will hold. After the transformation $I_i\mapsto I_i+V_i(\boldsymbol{J})$ we obtain $\{\boldsymbol{\psi},I_i\}=0$, while all other relations in (2) remain without changes, and the Hamiltonian takes the form

$$H=\hat{H}(\boldsymbol{I},\boldsymbol{J})+\mathcal{O}((\|\boldsymbol{I}\|+\|\boldsymbol{J}\|)^{l+1}). \tag{8}$$

Denote by $\bar{H}(\boldsymbol{I},\boldsymbol{J})$ the function which is obtained from H in (1) after dropping out the addendum $\mathcal{O}(\|\boldsymbol{I}\|^{l+1})$. Put

$$\bar{\lambda}_i(\boldsymbol{I},\boldsymbol{J})=\frac{\partial\bar{H}(\boldsymbol{I},\boldsymbol{J})}{\partial I_i},\quad i=1,...,m;$$

$$\bar{\lambda}_{m+j}(\boldsymbol{I},\boldsymbol{J})=\frac{\partial\bar{H}(\boldsymbol{I},\boldsymbol{J})}{\partial J_j}+\sum_{i=1}^{m}\bar{\lambda}_i(\boldsymbol{I},\boldsymbol{J})\frac{\partial V_i(\boldsymbol{J})}{\partial J_j},\quad j=1,...,k_0.$$

The system with the Hamiltonian $\bar{H}$ becomes

$$\dot{\boldsymbol{I}}=0;\quad \dot{\boldsymbol{J}}=0;\quad \dot{\varphi}_i=\bar{\lambda}_i(\boldsymbol{I},\boldsymbol{J}),\ i=1,...,m;$$

$$\dot{\boldsymbol{\psi}}=\sum_{j=1}^{k_0}\bar{\lambda}_{m+j}(\boldsymbol{I},\boldsymbol{J})\boldsymbol{\sigma}_j. \tag{9}$$

and obviously is integrable. Any common level $\boldsymbol{I} = \boldsymbol{I}_0 \neq 0,\ \boldsymbol{J} = \boldsymbol{J}_0$ is a r-dimensional ($r = m + k_0$) coisotropic invariant torus of the system (9). The motion on this torus is quasi-periodic with $r_1 \leq r$ rationally independent frequencies. The arithmetic properties of the frequencies are determined, in particular, by those of vectors $\boldsymbol{\sigma}_j$ and are "regulated" by functions $\bar{\lambda}_i(\boldsymbol{I}, \boldsymbol{J}),\ i = 1, ..., m + k_0$. The question whether the case of quasi-periodic motions with r rationally independent frequencies is in some sense typical depends to a large extent on non-degeneracy properties of the above functions. In section 3 these properties will be formulated in terms of functions $\bar{\lambda}_i(0, \boldsymbol{J})$, i.e. actually through

$$H^0(\boldsymbol{J}),\ \lambda_i(\boldsymbol{J}),\ \frac{\partial V(\boldsymbol{J})}{\partial J_j},\ j = 1, ..., k_0,\ i = 1, ..., m.$$

If the manifold of equilibria for the reduced system is already found then main technical difficulties appear in finding functions $V_i(J)$. It turns out that the above problem is got into the important class of mechanical systems which will be considered in the next section.

2 Mechanical systems with gyroscopic forces

Let $\mathcal{M}$ be a n-dimensional Riemannian manifold admitting free smooth isometric action of torus T^k and $\mathsf{p} : \mathcal{M} \mapsto \mathcal{N} = \mathcal{M}/\mathsf{T}^k$ be the natural projection of the corresponding principal T^k-fibre bundle. Consider the T^k-invariant mechanical system on $T^*\mathcal{M}$ with the total energy

$$H = T + U \circ pr := \frac{1}{2}(p|\mathcal{A}^{-1}(q)p) + U(q),$$

and a gyroscopic force Γ (we use notations of [7, 8]). This system may be regarded as a Hamiltonian one on the twisted cotangent bundle (M, ω^2), where $\mathsf{M} = T^*\mathcal{M}$, $\omega^2 = d\Lambda + pr^*\Gamma$, $d\Lambda$ is the standard symplectic structure on $T^*\mathcal{M}$ and $pr : \mathsf{M} \mapsto \mathcal{M}$ is the natural projection. Denote by Y_a the generator of the one-parameter subgroup action on $\mathcal{M}$, $\boldsymbol{a} \in \mathsf{t}^k$, and define a bilinear form $\mathcal{C}(\boldsymbol{a}, \boldsymbol{b}) = \Gamma(Y_a, Y_b)$ which is assumed to be nontrivial. Since in the sequel only the neighbourhood of r.e. will be investigated, suppose that the manifold $\mathcal{N}$ is simply connected. Then there exists the map $\boldsymbol{\mu} : \mathcal{M} \mapsto K^* := (\ker \mathcal{C})^*$ satisfying

$$d(\boldsymbol{\mu}|\boldsymbol{a}) = -\iota(Y_a)\Gamma, \quad \boldsymbol{a} \in K.$$

Thus we can define the momentun map $\boldsymbol{J}$ of K-action on M:

$$\iota(X_a)\omega^2 = -d(\boldsymbol{J}|\boldsymbol{a}) := -d(\boldsymbol{m} + \boldsymbol{\mu}|\boldsymbol{a}) \quad \boldsymbol{a} \in K.$$

(Recall that $\boldsymbol{m}$ denotes the momentum map of torus action on $T^*\mathcal{M}$.) Obviously, $d\boldsymbol{J} = \boldsymbol{\theta}$.

To reduce the above Hamiltonian system we use in [7] the map

$$\pi := \pi_0 \circ P_0 \times (\boldsymbol{m} + \mathcal{P}_K^* \boldsymbol{\mu}) : \mathsf{M} \mapsto T^*\mathcal{N} \times (\mathsf{t}^k)^*$$

which is the projection of the principal T^k-fibre bundle. Now we are going to concretize the reduction procedure and expose it in the form convenient for our purposes. Observe that a rather hard assumption from [7] about concordance of Γ and the Riemannian metric will not be used in our future reasoning.

2.1 Coordinates of the action-angle type near the relative equilibrium

Let $\tilde{\boldsymbol{q}} = (\tilde{q}_1, ..., \tilde{q}_{n-k})$ be coordinates in such a domain $\mathcal{N}' \subset \mathcal{N}$ that $\mathsf{p}^{-1}(\mathcal{N}') \sim \mathcal{N}' \times \mathsf{T}^k_\psi$, and $\tilde{\boldsymbol{p}} = (\tilde{p}_1, ..., \tilde{p}_{n-k})$ be canonically conjugate ones, so that the Liouville 1-form Λ on $T^*\mathcal{N}'$ is expressed as $\tilde{\boldsymbol{p}} d\tilde{\boldsymbol{q}}$. We also have

$$\omega^2 = d\left((\tilde{\boldsymbol{p}} d\tilde{\boldsymbol{q}}) + d(\boldsymbol{m}|d\boldsymbol{\psi})\right) + \Gamma,$$

$$H = T(\tilde{\boldsymbol{q}}; \tilde{\boldsymbol{p}}, \boldsymbol{m}) + U(\tilde{\boldsymbol{q}}),$$

where T is a quadratic form with respect to $\tilde{\boldsymbol{p}}, \boldsymbol{m}$. Let $L \subset \mathsf{t}^k$ be the subspace mentioned in the previous section. On the portion $\mathsf{p}^{-1}(\mathcal{N}')$ one can construct the connection form

$$\boldsymbol{\omega} = d\boldsymbol{\xi}(\tilde{\boldsymbol{q}}) + d\boldsymbol{\psi},$$

where $\boldsymbol{\xi} : \mathcal{N}' \mapsto L$ is a smooth map, in such a way that

$$\iota(Y_a) = \mathcal{C}(\boldsymbol{a}, \boldsymbol{\omega}), \quad \forall \boldsymbol{a} \in L.$$

Then

$$\Gamma = d\left(\boldsymbol{\zeta}(\tilde{\boldsymbol{q}}) d\tilde{\boldsymbol{q}} + (\boldsymbol{\mu}(\tilde{\boldsymbol{q}})|\mathcal{P}_K \boldsymbol{\omega})\right) + \frac{1}{2} \boldsymbol{C} \boldsymbol{\omega} \wedge \boldsymbol{\omega},$$

where $\boldsymbol{\zeta} : \mathcal{N}' \mapsto \mathbf{R}^{n-k}$ is a smooth map, and $\boldsymbol{C} : \mathsf{t}^k \mapsto (\mathsf{t}^k)^*$ is the linear operator determined by $(\boldsymbol{C}\boldsymbol{a}|\boldsymbol{b}) = \mathcal{C}(\boldsymbol{a}, \boldsymbol{b})$.

Identifying $L^\perp$ with K^* and $K^\perp$ with L^*, we have the decomposition $(\mathsf{t}^k)^* = K^* \oplus L^*$. Denote by $\bar{\boldsymbol{m}}$ the L^*-component of the map $\boldsymbol{m}$ and by $\boldsymbol{C}_L : L \mapsto L^*$ the operator corresponding to the bilinear form $\mathcal{C}|_L$. After the change of variables

$$\tilde{\boldsymbol{q}} = \breve{\boldsymbol{q}}, \quad \tilde{\boldsymbol{p}} = \breve{\boldsymbol{p}} + \frac{\partial}{\partial \breve{\boldsymbol{q}}} (\boldsymbol{m}|\boldsymbol{\xi}(\breve{\boldsymbol{q}})) - \boldsymbol{\zeta}(\breve{\boldsymbol{q}}),$$

$$\boldsymbol{\psi} = \breve{\boldsymbol{\psi}} - \boldsymbol{\xi}(\breve{\boldsymbol{q}}) - \boldsymbol{C}_L^{-1} \bar{\boldsymbol{m}},$$

we obtain

$$\omega^2 = d\breve{\boldsymbol{p}} \wedge d\breve{\boldsymbol{q}} + d\boldsymbol{J} \wedge \mathcal{P}_K d\breve{\boldsymbol{\psi}} - \frac{1}{2} d\bar{\boldsymbol{m}} \wedge \boldsymbol{C}_L^{-1} d\bar{\boldsymbol{m}} + \frac{1}{2} \boldsymbol{C} d\breve{\boldsymbol{\psi}} \wedge d\breve{\boldsymbol{\psi}}. \tag{10}$$

There exists the basis $\boldsymbol{\gamma}_1, ..., \boldsymbol{\gamma}_l, \boldsymbol{\eta}_1, ..., \boldsymbol{\eta}_l$ in $L \subset \mathsf{t}^k$ for which

$$\mathcal{C}(\boldsymbol{\gamma}_i, \boldsymbol{\gamma}_j) = 0, \quad \mathcal{C}(\boldsymbol{\gamma}_i, \boldsymbol{\eta}_j) = \delta_{ij} \quad \mathcal{C}(\boldsymbol{\eta}_i, \boldsymbol{\eta}_j) = 0$$

holds. Put $\bar{q}_i = (\boldsymbol{m}|\boldsymbol{\gamma}_i)$, $\bar{p}_j = (\boldsymbol{m}|\boldsymbol{\eta}_j)$ to get

$$\omega^2 = d\breve{\boldsymbol{p}} \wedge d\breve{\boldsymbol{q}} + d\bar{p} \wedge d\bar{\boldsymbol{q}} + d\boldsymbol{J} \wedge \mathcal{P}_K d\breve{\boldsymbol{\psi}} + \frac{1}{2} \boldsymbol{C} d\breve{\boldsymbol{\psi}} \wedge d\breve{\boldsymbol{\psi}}. \tag{11}$$

From this one can easily obtain the formulae for Poisson brackets:

$$\{\breve{q}_i, \breve{p}_j\} = \delta_{ij}; \quad \{\bar{q}_i, \bar{p}_j\} = \delta_{ij};$$

$$\{\breve{\boldsymbol{\psi}}, (\boldsymbol{J}|\boldsymbol{a})\} = \boldsymbol{a}, \ \boldsymbol{a} \in K; \quad \{\breve{\psi}_i, \breve{\psi}_j\} = \text{const}.$$

The brackets of all other of combinations pairwise coordinates vanish. Obviously, $\boldsymbol{u} = (\breve{\boldsymbol{p}}, \bar{\boldsymbol{p}}, \breve{\boldsymbol{q}}, \bar{\boldsymbol{q}})$ represents canonical coordinates on each symplectic leaf.

Proposition 1 *Let the T^k-invariant mechanical system with gyroscopic forces satisfy the conditions of Theorem 1. Then the functions $V_i(\boldsymbol{J}),\ i=1,\dots,m$ in formula (2) vanish.*

P r o o f. Let $\Lambda = \breve{\boldsymbol{p}}d\breve{\boldsymbol{q}} + \bar{\boldsymbol{p}}d\bar{\boldsymbol{q}}$ and $(\boldsymbol{I},\boldsymbol{\varphi})$ be the action-angle coordinates constructed in Theorem 1. The vector field $\Im_N dI_i$ in coordinates $(\boldsymbol{I},\boldsymbol{\varphi},\boldsymbol{J})$ becomes $\frac{\partial}{\partial\varphi_i}$. Then via the well known formula for the Lie derivative we have

$$\frac{\partial}{\partial\varphi_i}\Lambda = d\left(\Lambda(\frac{\partial}{\partial\varphi_i})\right) + \iota\left(\frac{\partial}{\partial\varphi_i}\right)d\Lambda.$$

Since $\iota(\Im dI_i)\omega^2 = -dI_i$, and the $\boldsymbol{\psi}$-component of the vector $\Im dI_i$ is of the form

$$\{\boldsymbol{\psi}, I_i\} = \sum_{j=1}^{m}\frac{\partial V_i(\boldsymbol{J})}{\partial J_j}\boldsymbol{\sigma}_j + \frac{\partial \boldsymbol{F}}{\partial\varphi_i},$$

then

$$\begin{aligned} -dI_i &= \frac{\partial}{\partial\varphi_i}\Lambda - d\left(\Lambda(\frac{\partial}{\partial\varphi_i})\right) - \{\boldsymbol{\psi}, V_i(\boldsymbol{J})\} - \\ &\quad -(d\boldsymbol{J}|\mathcal{P}_K\frac{\partial \boldsymbol{F}}{\partial\varphi_i}) + (\boldsymbol{C}\frac{\partial \boldsymbol{F}}{\partial\varphi_i}|d\boldsymbol{\psi}). \end{aligned}$$

Averaging with respect to $\boldsymbol{\varphi}$ leads to

$$dI_i = d\left(\frac{1}{(2\pi)^m}\int_{\mathrm{T}^m_{\varphi}}\Lambda(\frac{\partial}{\partial\varphi_i})d\boldsymbol{\varphi}\right) + \{\boldsymbol{\psi}, V_i(\boldsymbol{J})\}. \tag{12}$$

Observe that $\Lambda(\frac{\partial}{\partial\varphi_i})$ is the coefficient of $d\varphi_i$ in the 1-form Λ written in the coordinates $(\boldsymbol{I},\boldsymbol{\varphi},\boldsymbol{J})$. Since

$$d\Lambda|_{J=\mathrm{const}} = d\boldsymbol{I}\wedge d\boldsymbol{\varphi} = d(\boldsymbol{I}d\boldsymbol{\varphi}),$$

then averaging the above coefficient over the torus T^m_{φ} we must obtain the function of the form $I_i + G_i(\boldsymbol{J})$. On the other hand, from the proof of Theorem 1 it follows that Λ in the coordinates $(\boldsymbol{I},\boldsymbol{\varphi},\boldsymbol{J})$ becomes

$$\Lambda = \sum_{i=1}^{m}\left(\sum_{j=1}^{m}[a_{ij}(\boldsymbol{J},\boldsymbol{\varphi})\sqrt{2I_j} + u_j(\boldsymbol{J})]d\sum_{l=1}^{m}[b_{ij}(\boldsymbol{J},\boldsymbol{\varphi})\sqrt{2I_j} + v_j(\boldsymbol{J})]\right)$$

with appropriate functions a_{ij}, b_{ij}, u_j, v_j satisfying

$$\int_{\mathrm{T}^m_{\varphi}} a_{ij}d\boldsymbol{\varphi} = \int_{\mathrm{T}^m_{\varphi}} b_{ij}d\boldsymbol{\varphi} = 0.$$

Now it is clear that $G_i(\boldsymbol{J}) = 0$, and from (12) it follows that $\frac{\partial V_i(J)}{\partial J_j} = 0$. $\square$

2.2 The equilibria of a reduced system

Let $(\boldsymbol{y}, \boldsymbol{\psi})$ be coordinates of the direct product in a portion $\mathsf{p}^{-1}(\mathcal{N}')\sim\ \mathcal{N}' \times \mathsf{T}^k_\psi$ (for convenience we write $\boldsymbol{y}$ instead of $\tilde{\boldsymbol{q}}$). The kinetic energy takes the form

$$T = \frac{1}{2}\left((\tilde{\mathcal{A}}(\boldsymbol{y})\dot{\boldsymbol{y}}|\dot{\boldsymbol{y}}) + (\mathcal{B}(\boldsymbol{y})\dot{\boldsymbol{\psi}}|\dot{\boldsymbol{\psi}})\right) + (\mathcal{D}(\boldsymbol{y})\dot{\boldsymbol{y}}|\dot{\boldsymbol{\psi}}),$$

where

$$\tilde{\mathcal{A}}(\boldsymbol{y}) : T_y\mathcal{N}' \mapsto T^*_y\mathcal{N}', \quad \mathcal{B}(\boldsymbol{y}) : \mathsf{t}^k \mapsto (\mathsf{t}^k)^*, \quad \mathcal{D}(\boldsymbol{y}) : T_y\mathcal{N}' \mapsto (\mathsf{t}^k)^*$$

are operators that smoothly depend on $\boldsymbol{y}$.

Denote by $\hat{\boldsymbol{\omega}}$ the connection form naturally generated by the Riemannian metric on $\mathcal{M}$, so that the equality $\hat{\boldsymbol{\omega}}(\boldsymbol{\xi}) = 0$ is equivalent to the orthogonality condition of the vector $\boldsymbol{\xi}$ to the orbit of torus action. One can easily verify that $\hat{\boldsymbol{\omega}} = \ = d\boldsymbol{\psi} + \mathcal{E}(\boldsymbol{y})d\boldsymbol{y}$, where $\mathcal{E} = \mathcal{B}^{-1}\mathcal{D}$, and then

$$T = \frac{1}{2}\left((\hat{\mathcal{A}}(\boldsymbol{y})\dot{\boldsymbol{y}}|\dot{\boldsymbol{y}}) + (\mathcal{B}(\boldsymbol{y})\hat{\boldsymbol{\omega}}|\hat{\boldsymbol{\omega}})\right),$$

where $\hat{\mathcal{A}} = \tilde{\mathcal{A}} - \mathcal{D}^*\mathcal{B}^{-1}\mathcal{D}$. Having introduced the momenta

$$\hat{\boldsymbol{p}} = \hat{\mathcal{A}}\dot{\boldsymbol{y}}, \quad \boldsymbol{m} = \mathcal{B}\hat{\boldsymbol{\omega}},$$

we obtain the coordinates $(\hat{\boldsymbol{p}}, \boldsymbol{m}, \boldsymbol{y}, \boldsymbol{\psi})$ on $T^*\mathcal{N}'$ in which the Liouville form is represented as $\hat{\boldsymbol{p}}d\boldsymbol{y} + (\boldsymbol{m}|\hat{\boldsymbol{\omega}})$, and the Hamiltonian takes the form

$$H = \frac{1}{2}(\hat{\boldsymbol{p}}|\hat{\mathcal{A}}^{-1}(\boldsymbol{y})\hat{\boldsymbol{p}}) + \frac{1}{2}(\boldsymbol{m}|\mathcal{B}^{-1}(\boldsymbol{y})\boldsymbol{m}) + U(\boldsymbol{y}).$$

Observe that the standard momentum

$$\tilde{\boldsymbol{p}} = \frac{\partial T}{\partial \dot{\boldsymbol{y}}} \equiv \tilde{\mathcal{A}}(\boldsymbol{y})\dot{\boldsymbol{y}} + \mathcal{D}^*(\boldsymbol{y})\dot{\boldsymbol{\psi}}$$

is connected with $\hat{\boldsymbol{p}}$ by the relation

$$\hat{\boldsymbol{p}} = \tilde{\boldsymbol{p}} - \mathcal{E}^*\boldsymbol{m}.$$

Let $\imath : K \mapsto \mathsf{t}^k$ be an embedding. Define the family of projections

$$\{P^*_i(\boldsymbol{y}) : (\mathsf{t}^k)^* \mapsto (\mathsf{t}^k)^*\},\ i = 1, 2,$$

$$P^*_1(\boldsymbol{y}) = \mathcal{B}(\boldsymbol{y})\imath\mathcal{B}^{-1}_K(\boldsymbol{y})\imath^*, \quad P^*_2(\boldsymbol{y}) = \boldsymbol{Id} - P^*_1(\boldsymbol{y}),$$

where $\mathcal{B}_K(\boldsymbol{y}) := \imath^*\mathcal{B}(\boldsymbol{y})\imath : K \mapsto K^*$ is the symmetric positive operator.

Theorem 2 *For any $c \in K^*$ the point $(\hat{\boldsymbol{p}}_0, \boldsymbol{m}_0, \boldsymbol{y}_0)$ is the stationary one for $H|_{\boldsymbol{J}^{-1}(c)}$ iff*

$$\hat{\boldsymbol{p}}_0 = 0, \quad \boldsymbol{m}_0 = P^*_1(\boldsymbol{y}_0)\boldsymbol{m}_0 = \mathcal{B}(\boldsymbol{y}_0)\imath\mathcal{B}^{-1}_K(\boldsymbol{y}_0)(c - \boldsymbol{\mu}(\boldsymbol{y}_0)),$$

and $\boldsymbol{y}_0$ is a stationary point for the "effective potential"

$$U_c(\boldsymbol{y}) = \frac{1}{2}\left(c - \boldsymbol{\mu}(\boldsymbol{y})|\mathcal{B}^{-1}_K(\boldsymbol{y})(c - \boldsymbol{\mu}(\boldsymbol{y}))\right) + U(\boldsymbol{y}).$$

P r o o f. Since for any $\boldsymbol{y} \in \mathcal{N}'$ the subspaces

$$L_2^*(\boldsymbol{y}) = P_2^*(\boldsymbol{y})(\mathsf{t}^k)^*, \quad L_1^*(\boldsymbol{y}) = P_1^*(\boldsymbol{y})(\mathsf{t}^k)^*$$

are orthogonal with respect to the scalar product $(\cdot|\mathcal{B}^{-1}(\boldsymbol{y})\cdot)$, it follows that

$$\begin{aligned} H &= \frac{1}{2}(\hat{\boldsymbol{p}}|\hat{\mathcal{A}}^{-1}(\boldsymbol{y})\hat{\boldsymbol{p}}) + \frac{1}{2}(P_2^*(\boldsymbol{y})\boldsymbol{m}|\mathcal{B}^{-1}(\boldsymbol{y})P_2^*(\boldsymbol{y})\boldsymbol{m}) + \\ &\quad + \frac{1}{2}(\imath^*\boldsymbol{m}|\mathcal{B}_K^{-1}(\boldsymbol{y})\imath^*\boldsymbol{m} + U(\boldsymbol{y}). \end{aligned}$$

From the definition of $\boldsymbol{J}$ we obtain $\imath^*\boldsymbol{m} = \boldsymbol{J} - \boldsymbol{\mu}$. Next, observe that the map $P_2^*(\boldsymbol{y})|_{(\imath^*)^{-1}(c-\mu)}$ is one-to-one for any $\boldsymbol{y} \in \mathcal{N}'$. This implies that the condition $d(H|_{J^{-1}(c)}) = 0$ is equivalent to $\hat{\boldsymbol{p}} = 0$, $P_2^*(\boldsymbol{y})\boldsymbol{m} = 0$, $dU_c = 0$. $\square$

3 KAM-theory for coisotropic quasi-periodic motions near the relative equilibrium

Consider the system with the Hamiltonian (1). For $R_I \ll 1$ the addend $\mathcal{O}(\|\boldsymbol{I}\|^{l+1})$, which is the function of variables $\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{\varphi}$, plays a role of perturbation for the integrable Hamiltonian $\bar{H}$. In this situation we are going to apply the KAM-theory, in particular the results of [9], in order to establish the strict statement on the existence of coisotropic quasi-periodic motions. Generalizing our problem, we shall investigate the system with a Hamiltonian $\bar{H} + h$, where h is a "small" function which may depend not only on $\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{\varphi}$, but also on $\boldsymbol{\psi}$.

First of all, let us focus on the non-degeneracy condition of the map

$$\bar{\boldsymbol{\lambda}} : \mathcal{B}^m(0; R_I) \times \mathcal{B}^{k_0}(0; R_J) \mapsto \mathbf{R}^s, \; s := m + k_0,$$

where $\bar{\boldsymbol{\lambda}} = (\bar{\lambda}_1, ..., \bar{\lambda}_s)$, or of the map

$$\bar{\boldsymbol{\lambda}}' : \mathcal{B}^m(0; R_I) \times \mathcal{B}^{k_0}(0; R_J) \mapsto \mathbf{RP}^{s-1},$$

where $\bar{\boldsymbol{\lambda}}' = \bar{\lambda}_1 : \cdots : \bar{\lambda}_s$. The above condition must ensure that the image of the map $\bar{\lambda}$ or $\bar{\lambda}'$ can be stratified by unflattening curves in the sense of [12].

In the case when $R_I \ll R_J$ it is sufficient to impose the non-degeneracy condition on the map $\boldsymbol{\lambda}(\boldsymbol{J}) := \bar{\boldsymbol{\lambda}}(0, \boldsymbol{J})$, in which the first m components are determined by the functions $\lambda_i(\boldsymbol{J})$ in (1), and others are of the form

$$\lambda_{m+j}(\boldsymbol{J}) = \frac{\partial H^0(\boldsymbol{J})}{\partial J_j} + \sum_{i=1}^m \lambda_i(\boldsymbol{J}) \frac{\partial V_i(\boldsymbol{J})}{\partial J_j}.$$

Definition 1 *The map* $\boldsymbol{\lambda} : \mathcal{B}^{k_0}(0; R_J) \mapsto \mathbf{R}^s$ *will be called unflattening if there exists a surjective diffeomorphism*

$$\boldsymbol{w} : \{z \in [0, 1]\} \times \mathcal{B}^{k_0 - 1}(0; 1) \mapsto \mathcal{B}^{k_0}(0; R_J),$$

such that for some $\Delta > 0$ *the condition*

$$|\det\left(\frac{\partial}{\partial z}\boldsymbol{\lambda}\circ\boldsymbol{w};\frac{\partial^2}{\partial z^2}\boldsymbol{\lambda}\circ\boldsymbol{w};...;\frac{\partial^s}{\partial z^s}\boldsymbol{\lambda}\circ\boldsymbol{w}\right)|\geq\Delta>0. \tag{13}$$

is fulfilled in the domain of definition of $\boldsymbol{w}$*.*

To avoid technical complications we shall consider real analytic objects instead of smooth ones.

Theorem 3 *Suppose that the* T^k*-invariant system with the Hamiltonian* H *satisfies the conditions of Theorem 1 with* $l > s+3$*, the map* $\boldsymbol{\lambda}(\boldsymbol{J})$ *is unflattening, and for some* $\gamma > 0$ *the following inequalities are valid:*

$$\sum_{j=1}^{k_0}|(\boldsymbol{n},\boldsymbol{\sigma}_j)|>\gamma\left(\sum_{j=1}^{k}|n_i|\right)^{-r},$$

$\forall\boldsymbol{n}=(n_1,...,n_k)\in\mathbf{Z}^r\setminus\{0\},\ r=m+k_0$*. Let* ε *be an arbitrary positive number.*

Then there exists $\delta(\varepsilon)>0$ *such that for* $R_I<\delta(\varepsilon)$ *and* $|h|<R_I^{s+4}$ *there exists a set* $\mathcal{Q}\subset\pi^{-1}(\mathcal{V}(x_0))$ *with the following properties:*

1.
$$\mathrm{mes}\mathcal{Q}>(1-\varepsilon)\mathrm{mes}[\pi^{-1}(\mathcal{V}(x_0))].$$

2. *The motion of each point belonging to* $\mathcal{Q}$ *under the action of the flow generated by the Hamiltonian vector field* $\Im d(\bar{H}+h)$ *is coisotropic quasi-periodic with* r *rationally independent frequencies.*

P r o o f. To apply the main theorem of [9] we put

$$G=\{\boldsymbol{z}\in\mathbf{C}^s:\mathrm{Re}\boldsymbol{z}\in\mathcal{B}^m(0;R_I)\times\mathcal{B}^{k_0}(0;R_J),\ |\mathrm{Im}\boldsymbol{z}|<\varrho\},$$

where ϱ is a sufficiently small positive number, and consider $\bar{H}$ in (1) to be the unperturbed Hamiltonian. Next, we fix R_J, whereas the number R_I will be used to controll the value of the perturbation $M\sim R_I^{s+4}$. Following the proof in [9] we construct the sequence of sets

$$G\supset G_1\supset\cdots G_j\supset\cdots$$

and the sequence of diffeomorphisms

$$\Phi_j:\mathrm{Re}G_j\times\mathsf{T}^r\mapsto\mathrm{Re}G\times\mathsf{T}^r,$$

such that

$$(\bar{H}+h)\circ\Phi_{j+1}(\boldsymbol{I},\boldsymbol{J},\boldsymbol{\varphi},\boldsymbol{\psi})=H_{j+1}(\boldsymbol{I},\boldsymbol{J})+h_{j+1}(\boldsymbol{I},\boldsymbol{J},\boldsymbol{\varphi},\boldsymbol{\psi}).$$

We can pick out such small positive δ and α that the inequalities

$$|h_j(x)|<R_I^{(s+3)(1+\alpha)^j},\quad|x-\Phi_j(x)|<\delta,\quad|\boldsymbol{Id}-\frac{\partial\Phi_j(x)}{\partial x}|<\delta \tag{14}$$

hold true for all $R_I \in (0, \delta]$ and $x \in \mathrm{Re}G_j$.

It turns out that the required set can be correctly defined as $\mathcal{Q} := \Phi_\infty(\mathrm{Re}G_\infty \times \mathsf{T}^k)$. We use the technique developed in [9] to estimate the Lebesgue measure for the intersection of the set $\mathrm{Re}G_\infty$ with each unflattening curve given by

$$\boldsymbol{I} = \boldsymbol{I}_0, \quad \boldsymbol{J} = \boldsymbol{w}(z, \boldsymbol{c}),$$

where $\boldsymbol{c} \in \mathcal{B}^{k_0-1}(0; 1)$, $\boldsymbol{I}_0 \in \mathcal{B}^m(0; R_I)$. Namely, one can show that for arbitrary

$$\eta \in (0, 1), \quad \boldsymbol{c} \in \mathcal{B}^{k_0-1}(0; 1-\eta) \quad \text{and} \quad \boldsymbol{I} \in \mathcal{B}^m(0; (1-\eta)R_I)$$

there exists the limit

$$\mathrm{mes}\{z \in [\eta, 1-\eta] : (\boldsymbol{w}(z, \boldsymbol{c}), \boldsymbol{I}) \in \mathrm{Re}G_\infty\} \to \eta(1-\eta), \quad R_I \to 0,$$

and this implies that

$$\mathrm{mes}\,\mathrm{Re}G_\infty \to \mathrm{mes}\,\mathrm{Re}G, \quad R_I \to 0.$$

□

4 Coisotropic quasi-periodic motions in a system of constrained rotors

Consider the mechanical system consisting of three axisymmetric rotors which are constrained in such a way that: 1) each rotor can rotate around its symmetry axis, which passes through rotor's center of mass; 2) the total senter of mass of constrained rotors admits rotations in Oyz-plane about x-axis.

The configuration space of the above system is a torus T^4. Denote by θ the angle between the position vector of center of mass and vector $-\boldsymbol{e}_z$, and by ψ_i the angle which measures rotation of the i-th rotor about its symmetry axis. The configuration space will now be considered as a trivial T^3-fibre bundle over circle S^1. Suppose that besides the force of gravity also the gyroscopic force acts upon our system, the corresponding 2-form being as follows:

$$\Gamma = \sum_{i=1}^{3} g_i(\theta) d\theta \wedge d\psi_i + \sigma_1 d\psi_2 \wedge d\psi_3 + \sigma_2 d\psi_3 \wedge d\psi_1 + \sigma_3 d\psi_1 \wedge d\psi_2.$$

Here $g_i : \mathsf{S}^1 \mapsto \mathbf{R}$ are some smooth functions, $\sigma_i \in \mathbf{R}$.

The kinetic energy of the above system is of the form

$$\frac{1}{2}\tilde{A}\dot{\theta}^2 + \sum_{i=1}^{3}(B_i a_i \dot{\theta}\dot{\psi}_i + \frac{1}{2}B_i\dot{\psi}_i^2),$$

where $\tilde{A}$ is system's moment of inertia with respect to x-axis, B_i is the i-th rotor's moment of inertia with respect to its symmetry axis, a_i is the direction cosine of this symmetry axis with respect to x-axis.

The potential of the system is

$$U(\theta) = -\rho \cos\theta$$

with some $\rho > 0$. Having introduced momenta

$$\tilde{p} = \tilde{A}\dot{\theta} + \sum_{i=1}^{3} B_i a_i \dot{\psi}_i, \quad p_i = B_i \psi_i + B_i a_i \dot{\theta},$$

one obtaines the Hamiltonian of the system

$$H = \frac{1}{2\hat{A}}(\tilde{p} - \sum_{i=1}^{3} a_i p_i)^2 + \sum_{i=1}^{3} \frac{1}{2B_i} p_i^2 - \rho\cos\theta,$$

where $\hat{A} = \tilde{A} - \sum_{j=1}^{3} B_j a_j^2$. The Hamiltonian and the form of the gyroscopic force are invariant with respect to the action of torus

$$\mathsf{T}^3_\psi = \{\boldsymbol{\psi} = (\psi_1, \psi_2, \psi_3) | \mathrm{mod}\, 2\pi\}.$$

Using the approach developed in the previous section we shall show how one can eliminate the cyclic coordinates ψ_1, ψ_2, ψ_3 in the case of the symplectic structure

$$\omega^2 = d\tilde{p} \wedge d\theta + \sum_{i=1}^{3} dp_i \wedge d\psi_i + \Gamma.$$

For this purpose we shall find the Casimir function of r.P.s. and construct the canonical coordinates on each symplectic leaf.

Let $(\cdot,\cdot)$ be the standard scalar product in t^3 naturally associated with coordinates ψ_1, ψ_2, ψ_3 on T^3_ψ. We identify t^3 with $(\mathsf{t}^3)^*$ by means of the above scalar product. The matrix of the operator $\mathcal{B}$ defined in the previous section becomes

$$\mathbf{B} = \mathrm{diag}(B_1, B_2, B_3),$$

and the matrix of the 2-cocycle $\mathcal{C}$ is of the form

$$\mathbf{C} = \begin{pmatrix} 0 & \sigma_3 & -\sigma_2 \\ -\sigma_3 & 0 & \sigma_1 \\ \sigma_2 & -\sigma_1 & 0 \end{pmatrix}.$$

Obviously, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3) \in \ker \mathcal{C}$. Besides the standard scalar product we denote the $\mathbf{B}$-scalar product as $(\mathbf{B}\cdot,\cdot)$. Now introduce in t^3 a $\mathbf{B}$-orthogonal basis

$$\boldsymbol{\epsilon}_1 = \frac{1}{\sqrt{(\mathbf{B}\boldsymbol{\sigma}, \boldsymbol{\sigma})}}, \quad \boldsymbol{\epsilon}_2, \quad \boldsymbol{\epsilon}_3,$$

which satisfies the conditions

$$(\mathbf{B}\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_1) = 1; \quad (\mathbf{B}\boldsymbol{\epsilon}_i, \boldsymbol{\epsilon}_i) = 1/\nu, \ i = 2, 3; \quad \mathcal{C}(\boldsymbol{\epsilon}_2, \boldsymbol{\epsilon}_3) = 1$$

for appropriate $\nu > 0$ and a dual one

$$\boldsymbol{\epsilon}_1^* = \mathbf{B}\boldsymbol{\epsilon}_1, \quad \boldsymbol{\epsilon}_i^* = \nu\mathbf{B}\boldsymbol{\epsilon}_i, \ i = 2, 3.$$

Denote by L a subspace spanned by $\boldsymbol{\epsilon}_2, \boldsymbol{\epsilon}_3$. According to our approach we define a connection form

$$\boldsymbol{\omega} = \sum_{i=1}^{3} \omega_i \boldsymbol{\epsilon}_i$$

in such a way that

$$\iota(Y_a) = \mathcal{C}(\boldsymbol{a}, \boldsymbol{\omega}) \quad \forall \boldsymbol{a} \in L,$$

where Y_a is the vector field which generates the action of a one-parameter subgroup of T^3 corresponding to $\boldsymbol{a}$, so that $\iota(Y_a)d\boldsymbol{\psi} = \boldsymbol{a}$. It is easy to see that

$$\begin{aligned}
\omega_1 &= (\boldsymbol{\epsilon}_1^*, d\boldsymbol{\psi}); \\
\omega_2 &= (\boldsymbol{g}(\theta), \boldsymbol{\epsilon}_3)d\theta - \mathcal{C}(\boldsymbol{\epsilon}_3, d\boldsymbol{\psi}) \\
\omega_3 &= -(\boldsymbol{g}(\theta), \boldsymbol{\epsilon}_2)d\theta + \mathcal{C}(\boldsymbol{\epsilon}_2, d\boldsymbol{\psi}),
\end{aligned}$$

where $\boldsymbol{g}(\theta) = (g_1(\theta), g_2(\theta), g_3(\theta))$, and thus

$$\boldsymbol{\omega} = d\boldsymbol{\xi}(\theta) + d\boldsymbol{\psi},$$

where

$$d\boldsymbol{\xi}(\theta) = ((\boldsymbol{g}(\theta), \boldsymbol{\epsilon}_3)\boldsymbol{\epsilon}_2 - (\boldsymbol{g}(\theta), \boldsymbol{\epsilon}_2)\boldsymbol{\epsilon}_3)d\theta.$$

Put

$$\boldsymbol{p} = (p_1, p_2, p_3); \quad m_i = (\boldsymbol{p}, \boldsymbol{\epsilon}_i), \ i = 1, 2, 3; \quad \boldsymbol{m} = \sum_{i=1}^{3} m_i \boldsymbol{\epsilon}_i^*.$$

It is clear that m_i is a component of the momentum map of T^3-action on $T^*\mathsf{T}^4$ with respect to the standard symplectic structure $d(\tilde{p}d\theta + \sum_{i=1}^{3} p_i d\psi_i)$ and basis $\{\boldsymbol{\epsilon}_i^*\}_{i=1}^{3}$ of the dual space $(\mathsf{t}^3)^*$. Now we have

$$\iota(X_{\epsilon_1})\omega^2 = -(dm_1 + (\boldsymbol{g}(\theta), \boldsymbol{\epsilon}_1)d\theta),$$

from whence

$$J = m_1 + \int (\boldsymbol{g}(\theta), \boldsymbol{\epsilon}_1)d\theta$$

is the local Casimir function. After the change of variables

$$p_\theta = \tilde{p} - m_2(\boldsymbol{g}(\theta), \boldsymbol{\epsilon}_3) + m_3(\boldsymbol{g}(\theta), \boldsymbol{\epsilon}_2), \ \bar{q} = m_2, \ \bar{p} = m_3,$$

$$d\breve{\boldsymbol{\psi}} = d\boldsymbol{\psi} + [(\boldsymbol{g}(\theta), \boldsymbol{\epsilon}_3)\boldsymbol{\epsilon}_2 - (\boldsymbol{g}(\theta), \boldsymbol{\epsilon}_2)\boldsymbol{\epsilon}_3]d\theta + \boldsymbol{\epsilon}_2 dm_3 - \boldsymbol{\epsilon}_3 dm_2$$

the symplectic structure becomes

$$\omega^2 = d(p_\theta d\theta + \bar{p}d\bar{q} + J(\boldsymbol{\epsilon}_1^*, d\breve{\boldsymbol{\psi}})) + \sigma_1 d\breve{\psi}_2 \wedge d\breve{\psi}_3 + cycle,$$

and we get following formulae for the Poisson brackets of coordinates $p_\theta, \theta, \bar{q}, \bar{p}, J$:

$$\{\theta, p_\theta\} = 1, \qquad \{\bar{q}, \bar{p}\} = 1.$$

The brackets of all other combinations of pairwise coordinates vanish. So, $p_\theta, \theta, \bar{q}, \bar{p}, J$ represent canonical coordinates on each symplectic leaf $J = \text{const}$.

Since

$$\sum_{i=1}^{3} \frac{1}{B_i} p_i^2 = (\mathbf{B}^{-1} \sum_{i=1}^{3} m_i \boldsymbol{\epsilon}_i^*, \sum_{i=1}^{3} m_i \boldsymbol{\epsilon}_i^*) = m_1^2 + \nu(m_2^2 + m_3^2),$$

the Hamiltonian in above coordinates becomes

$$\begin{aligned} H &= \frac{1}{2\hat{A}} \left(p_\theta + \bar{q} f(\theta) + \bar{p} h(\theta) + (J - \mu(\theta)) b\right)^2 + \\ &+ \frac{1}{2}\nu \left(\bar{q}^2 + \bar{p}^2\right) + \frac{1}{2}\left(J - \mu(\theta)\right)^2 - \rho\cos\theta, \end{aligned} \tag{15}$$

where

$$f(\theta) = (\boldsymbol{a}, \boldsymbol{\epsilon}_2) + (\boldsymbol{g}(\theta), \boldsymbol{\epsilon}_3), \quad h(\theta) = (\boldsymbol{a}, \boldsymbol{\epsilon}_3) - (\boldsymbol{g}(\theta), \boldsymbol{\epsilon}_2),$$

$$\boldsymbol{a} = (a_1, a_2, a_3), \quad b = (\boldsymbol{a}, \boldsymbol{\epsilon}_1), \quad \mu(\theta) = \int (\boldsymbol{g}(\theta), \boldsymbol{\epsilon}_1) d\theta.$$

Now one can show that the equilibria of the reduced system are defined from

$$\rho \sin\theta - (J - \mu(\theta))\mu'(\theta) = 0; \tag{16}$$

$$\bar{q} = \bar{p} = 0; \quad p_\theta = (\mu(\theta) - J)b.$$

Now it is necessary to ascertain the conditions which guarantee that the system linearized at equilibrium is stable and possesses different eigenfrequencies. To avoid technical complications we shall consider the case where

$$\hat{A} = 1, \quad \boldsymbol{g}(\theta) = \boldsymbol{g} = \mathbf{const}, \quad (\boldsymbol{g}, \boldsymbol{\epsilon}_1) := g \neq 0.$$

Then

$$f(\theta) = f = \text{const}, \quad h(\theta) = h = \text{const}, \quad \mu(\theta) = g \cdot \theta,$$

and the first equation in (16) becomes

$$\rho \sin\theta - (J - g\theta)g = 0, \quad (\theta \in [0, 2\pi)). \tag{17}$$

It possesses at most three roots in the segment $[0, 2\pi)$. Let $\theta^* = \theta^*(J)$ be one of the above roots. The corresponding value of momentum is

$$p_{\theta^*} = (g\theta^* - J)b = \frac{b\rho}{g} \sin\theta^*.$$

One can show that for any fixed J and $p_\theta = p_{\theta^*}$, $\theta = \theta^*$, $\bar{q} = 0$, $\bar{p} = 0$ the characteristic polynomial of the linearized system is of the form

$$\lambda^4 + (g^2 + \rho\cos\theta^* + \nu(f^2 + g^2) + \nu^2)\lambda^2 + (g^2 + \rho\cos\theta^*)\nu^2,$$

nd the square of its roots is

$$\begin{aligned} \lambda_{1,2}^2 &= -\frac{1}{2}\left(g^2 + \rho\cos\theta^* + \nu(f^2 + g^2) + \nu^2\right) \pm \\ &\pm \frac{1}{2}\sqrt{(g^2 + \rho\cos\theta^* + \nu(f^2 + h^2) + \nu^2)^2 - 4(g^2 + \rho\cos\theta^*)\nu^2}. \end{aligned}$$

These quantities will be negative and different if

$$g^2 + \rho \cos \theta^* > 0. \tag{18}$$

Next, since

$$H^0(J) = \frac{1}{2}(J - g\theta^*)^2 - \rho \cos \theta^*,$$

then taking into account (17) we obtain

$$\lambda_3(J) = \frac{\partial H^0(J)}{\partial J} = (J - g\theta^*) = \frac{\rho}{g} \sin \theta^*.$$

To apply Theorem 2 we need to verify that (13) holds true. One can pick out a diffeomorphism $w : [0,1] \mapsto \mathbf{R}$ in such a way that

$$g^2 + \rho \cos \theta^*(w(z)) + \nu(f^2 + h^2) - \nu^2 = z + a$$

for some $a \in \mathbf{R}$. Then

$$\lambda_{1,2}^2(z) = \frac{1}{2}(-(z+b) \pm \sqrt{(z+a)^2 + c}),$$

$$\lambda_3(z) = \frac{\rho}{g}\sqrt{1 - (z+e)^2/\rho^2},$$

where $b, c, e \in \mathbf{R}$ are constants. Obviously the Wronskian of functions $\frac{d}{dz}\lambda_i(z),\ i = 1,2,3,$ does not equal identically to zero. Thus we arrive at the following

Conclusion If the condition (18) is valid for $J \in [0, R_J]$, and for some $\gamma > 0$ the inequalities

$$\sum_{j=1}^{3} |n_j \sigma_j)| > \gamma \left(\sum_{j=1}^{3} |n_i| \right)^{-5} \qquad \forall (n_1, ..., n_k) \in \mathbf{Z}^3 \setminus \{0\},$$

are satisfied then the system under consideration possesses coisotropic quasi-periodic motions with five rationally independent frequencies.

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