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Symmetry and Nonlocal Ansatzes for Nonlinear Heat Equations

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Abstract

Operators of nonlocal symmetry are used to construct exact solutions of nonlinear heat equations

In [1] the idea of constructing nonlocal symmetry of differential equations was proposed. By using this symmetry, we have suggested a method for finding new classes of ansatzes reducing nonlinear wave equations to systems of ordinary differential equations [2]. In the present paper, we apply this method to the nonlinear heat equation.

I Let us consider the equation

$$u_{xx} = F(u_t). \tag{1}$$

where $F(u_t)$ is some smooth function.
This equation is connected with the equation

$$w_t - (c(w)w_x)_x = 0. \tag{2}$$

In fact, the variable $w \equiv u_{xx}$ satisfies eq.(2), when

$$c(w) = \Big[F^{-1}(w)\Big]_w. \tag{3}$$

For convenience, we study the symmetry of eq.(1). The following system

$$\begin{aligned} &v_2^1 + v_3^1 v^2 = v_1^2 + v_3^2 v^1, \\ &v_2^2 + v_3^2 v^2 = F(v^1), \end{aligned} \tag{4}$$

where $t \equiv x_1$, $x \equiv x_2$, $u \equiv x_3$, $\dfrac{\partial u}{\partial t} = v^1, \dfrac{\partial u}{\partial x} = v^2$ $v_k^i \equiv \dfrac{\partial v^i}{\partial x_k}$, corresponds to eq.(1) if we use the approach suggested in [2]. The point symmetry of system (4) may be usefull in looking for solutions of equation (1) using a reduction method. Thus, we seek the invariance algebra of system (4) in the class of operators

$$X = \xi^1\partial_{x_1} + \xi^2\partial_{x_2} + \xi^3\partial_{x_3} + \eta^1\partial_{v_1} + \eta^2\partial_{v_2}, \tag{5}$$

where ξ^1, ξ^2, ξ^3, η^1, η^2 are functions depending on the variables x_1, x_2, x_3 v^1, v^2.

Then the following theorem can be proved with help of the Lie algorithm.

Theorem. *System (4) is invariant with respect to the algebra with basis elements*

$$\begin{aligned}
&P_1 = \partial_{x_1}, \quad P_2 = \partial_{x_2}, \quad P_3 = \partial_{x_3},\\
&D = 2x_1\partial_{x_1} + x_2\partial_{x_2} + 2x_3\partial_{x_3} + v^2\partial_{v^2}, \qquad (6)\\
&Q_1 = x_2\partial_{x_3} + \partial_{v^2},
\end{aligned}$$

$$Q_2 = -x_1\partial_{x_1} + v^2\partial_{x_2} + v^1\partial_{v_1}, \qquad (7)$$

if

$$F = \frac{1}{\alpha \ln v^1}. \qquad (8)$$

We consider a subalgebra generated by the operators $\{P_3, Q_2\}$. The ansatz corresponding to these operators is as follows

$$\begin{aligned}
&v^1 = \frac{\varphi_1(v^2)}{x_1},\\
&v^2 = \frac{x_2}{(\varphi_2(v^2) - \ln x_1)}.
\end{aligned} \qquad (9)$$

Substituting (9) into (4), we obtain the system of ordinary differential equations

$$\begin{aligned}
&\ln\varphi_1 - \varphi_2 = v^2\frac{d\varphi_2}{dv^2},\\
&\frac{d\varphi_1}{dv^2} = v^2.
\end{aligned} \qquad (10)$$

The general solution of system (10) has the form

$$\begin{aligned}
&\varphi_1 = \frac{1}{2}\left((v^2)^2 + C\right),\\
&\varphi_2 = \ln\frac{((v^2)^2 + C)}{2e^2} + \frac{C_1}{v^2} + \frac{2C}{v^2}\int\frac{dv^2}{(v^2)^2 + C}.
\end{aligned} \qquad (11)$$

Thus we obtain the system

$$\begin{aligned}
&u_t = \frac{(u_x)^2 + C}{2t},\\
&u_x = \frac{2x}{\ln\dfrac{(u_x)^2 + C}{2te^2} + \dfrac{C_1}{u_x} + \dfrac{2\sqrt{C}}{u_x}\arctan\dfrac{u_x}{\sqrt{C}}},
\end{aligned} \qquad (12)$$

when $C > 0$, and

$$\begin{aligned}
&u_t = \frac{(u_x)^2 + C}{2t},\\
&u_x = \frac{2x}{\ln\dfrac{(u_x)^2 + C}{2te^2} + \dfrac{C_1}{u_x} + \dfrac{\sqrt{-C}}{u_x}\ln\dfrac{u_x - \sqrt{-C}}{u_x + \sqrt{-C}}},
\end{aligned} \qquad (13)$$

when $C < 0$. It is necessary to integrate system (12) or (13) to construct the solution of equation (1). But using the connection between equations (1) and (2), it is easy to obtain the solution of equation (2) with $c(w) = -\dfrac{\exp\frac{1}{w}}{w^2}$ in the form

$$\exp\frac{1}{w} = \frac{(\theta)^2 + C}{2t}, \tag{14}$$

$$\theta = \frac{2x}{\ln\dfrac{(\theta)^2 + C}{2te^2} + \dfrac{C_1}{\theta} + \dfrac{2\sqrt{C}}{\theta}\arctan\dfrac{\theta}{\sqrt{C}}}, \tag{15}$$

when $C > 0$, and

$$\exp\frac{1}{w} = \frac{(\theta)^2 + C}{2t}, \tag{16}$$

$$\theta = \frac{2x}{\ln\dfrac{(\theta)^2 + C}{2te^2} + \dfrac{C_1}{\theta} + \dfrac{\sqrt{-C}}{\theta}\ln\dfrac{\theta - \sqrt{-C}}{\theta + \sqrt{-C}}}, \tag{17}$$

where $C < 0$, θ is the solution of (15) or (17). Thus formulae (14) and (16) give two families of solutions for the nonlinear heat equation (2).

II Let us consider the equation

$$u_t - u_{xx} = H(u), \tag{18}$$

where $H(u)$ is some smooth function.

The following system

$$\begin{aligned} &v^1_t + v^1_3 v^2 = v^2_3 v^1, \\ &v^2 - v^1_3 v^1 = H(x_3), \end{aligned} \tag{19}$$

where $u \equiv x_3$, $\dfrac{\partial u}{\partial x} \equiv v^1$, $\dfrac{\partial u}{\partial t} \equiv v^2$, corresponds to Eq.(18) if we use the approach suggested in [4]

Theorem 1 *System (19) is Q–conditionally invariant with respect to the operator*

$$Q = \partial_{x_3} + 2F\exp\left(-F^2\right)v^1\partial_{v^1} + \left(2F\exp\left(-F^2\right)v^2 + \frac{\exp\left(-F^2\right)v^2 - 1}{F}\right)\partial_{v^2} \tag{20}$$

if

$$H(x_3) = \exp\left(F^2(x_3)\right),$$

where $F(x_3) = \Phi^{-1}(x_3)$, $\Phi(x_3) = \int \exp\left((x_3)^2\right)dx_3$.

Proof. We use the criterion of Q–conditional invariance. Thus we have

$$\begin{aligned} &\widetilde{Q}(v^2 - v^1_3 v^1 - \exp\left(F^2(x_3)\right)) = \\ &\quad 2F\exp\left(-F^2\right)v^2 + \left(\exp\left(-F^2\right)v^2 - 1\right)/F - \\ &\quad v^1\left(2F\exp\left(-F^2\right)v^1 - 4F^2\exp\left(-2F^2\right)v^1 + 2F\exp\left(-F^2\right)v^1_3\right) - \\ &\quad 2v^1_3 F\exp\left(-F^2\right)v^1 - 2F, \end{aligned} \tag{21}$$

where $\widetilde{Q}$ is the prolongation of the operator Q.

Taking into account

$$v^2 = 2F\exp(-F^2)(v^1)^2 + \exp(F^2),$$

$$v_3^1 = 2F\exp(-F^2)v^1,$$

$$v_3^2 = 2F\exp(-F^2)v^2 + \left(\exp(-F^2)v^2 - 1\right)/F$$

we obtain

$$\begin{aligned}\widetilde{Q}(v^2 - v_3^1 v^1 - \exp(F^2(x_3))) = \\ &2F\exp(-F^2)\Big(2F\exp(-F^2)(v^1)^2 + \exp(F^2)\Big) + \\ &\Big(\exp(-F^2)\Big(2F\exp(-F^2)(v^1)^2 + \exp(F^2)\Big) - 1\Big)/F - \\ &v^1\left(2\exp(-2F^2)v^1 - 4F^2\exp(-F^2)v^1 + 4F^2\exp(-2F^2)v^1\right) - \\ &4F^2\exp(-2F^2)(v^1)^2 - 2F \equiv 0.\end{aligned}$$

Similarly we receive

$$\widetilde{Q}\left(v_t^1 + v_3^1 v^2 - v_3^2 v^1\right) \equiv 0. \tag{22}$$

Q.E.D. The operator (3) generates the ansatz

$$\begin{aligned} v^1 &= \exp(F^2)\varphi_1(t), \\ v^2 &= \exp(F^2)(2F\varphi_2(t) + 1), \end{aligned} \tag{23}$$

where φ_1, φ_2 are unknown functions.

Substitution of (6) into (2) yields the system of two ordinary differential equations for φ_1, φ_2

$$\begin{aligned} d\varphi_1/dt &= 2\varphi_1\varphi_2, \\ \varphi_2 &= \varphi_1^2, \end{aligned} \tag{24}$$

whose general solution has the form

$$\begin{aligned} \varphi_1 &= 1/\sqrt{C - 4t}, \\ \varphi_2 &= 1/(C - 4t). \end{aligned} \tag{25}$$

Integrating the overdetermined but compatible system

$$\begin{aligned} u_x &= \exp(F^2(u))/\sqrt{C - 4t}, \\ u_t &= \exp(F^2)\left(2F/(C - 4t) + 1\right), \end{aligned} \tag{26}$$

we get the exact solution of the nonlinear heat equation with the function $H(u) = \exp\left(F^2(u)\right)$

$$u = \Phi\left(\frac{\pm 6x - \left(\sqrt{C-4t}\right)^3 + C_1}{6\sqrt{C-4t}}\right), \tag{27}$$

where $\Phi(z) = \int \exp z^2 dz$. The maximal invariance algebra of the equation

$$u_t - u_{xx} = \exp\left(F^2(u)\right) \tag{28}$$

is a 2–dimensional Lie algebra, whose basis elements are given by the formulae

$$P_x = \partial_x, \qquad P_t = \partial_t.$$

It is obvious that the solution (10) is not an invariant solution.

III Finally, we consider the equation

$$\frac{\partial^2 u}{\partial x_0^2} = F\left(\frac{\partial^2 u}{\partial x_o \partial x_1}, \frac{\partial^2 u}{\partial x_1^2}\right), \tag{29}$$

where F is an arbitrary smooth function.

One can obtain the Monge-Ampere equation setting $F = \left(\frac{\partial^2 u}{\partial x_0 \partial x_1}\right)^2 / \frac{\partial^2 u}{\partial x_1^2}$. Using the invariance of equation (29) under the operators $\partial_u, x_0\partial_u, x_1\partial_u$, we write it in the form of the following system.

$$\begin{aligned} &\frac{\partial v^1}{\partial x_1} = \frac{\partial v^2}{\partial x_0}, \\ &\frac{\partial v^3}{\partial x_0} = \frac{\partial v^2}{\partial x_1}, \\ &v^1 = F(v^2, v^3), \end{aligned} \tag{30}$$

where $\frac{\partial^2 u}{\partial x_0^2} \equiv v^1$, $\frac{\partial^2 u}{\partial x_0 \partial x_1} \equiv v^2$, $\frac{\partial^2 u}{\partial x_1^2} \equiv v^3$.

Theorem 2 *System (30) is invariant with respect to the continuous group of transformations with the infinitesimal operator*

$$X = \xi^0(v^1, v^2, v^3)\partial_{x_0} + \xi^1(v^1, v^2, v^3)\partial_{x_1}, \tag{31}$$

if ξ^0, ξ^1 *satisfy the system of equations*

$$\begin{aligned} &\frac{\partial \xi^0}{\partial x_0} = \frac{\partial \xi^0}{\partial x_1} = \frac{\partial \xi^1}{\partial x_0} = \frac{\partial \xi^1}{\partial x_1} = 0, \\ &\xi^1_1 F_1 + \xi^1_2 - \xi^0_1 F_2 - \xi^0_3 = 0, \\ &\xi^0_2 F_2 = \xi^0_3 F_1 + \xi^1_1 F_2 + \xi^1_3, \end{aligned} \tag{32}$$

where $\xi^k_a \equiv \frac{\partial \xi^k}{\partial v^a}$, $F_a \equiv \frac{\partial F}{\partial v^{a+1}}$, $k = 0, 1;\ a = 1, 2, 3.$

With the help of this operator, one can construct ansatzes reducing equation (29) to the system of three ordinary differential equations for three unknown functions.

The finite transformations

$$\tilde{x_0} = x_0 + a\xi^0, \qquad \tilde{x_1} = x_1 + a\xi^1 \tag{33}$$

correspond to operator (31). In the case of Lie-Bäcklund symmetry, one can construct finite transformations in a closed form for point and contact symmetries only. We note that although ξ^0 and ξ^1 depend on the second–order derivatives $\dfrac{\partial^2 u}{\partial x_0^2}$, $\dfrac{\partial^2 u}{\partial x_0 \partial x_1}$, $\dfrac{\partial^2 u}{\partial x_1^2}$ in terms of initial variables, we are able to construct the finite transformations (33) corresponding to operator (31). Moreover, these transformations can be used for generating new solutions. For example, we shall take $F = \sin\left(\dfrac{\partial^2 u}{\partial x_1^2}\right)$. In this case, one of the solutions of system (32) is

$$\begin{aligned} \xi^0 &= \frac{C}{2}(v^1)^2 - C\cos v^2 + D, \\ \xi^1 &= C v^1 \sin v^2 + D v^1, \end{aligned} \tag{34}$$

where C, D are constants. We start from solution $u = \frac{x_0 x_1^2}{2} - \sin x_0$ of the equation (29). Then

$$v^0 = \sin x_0, \qquad v^1 = x_1, \qquad v^2 = x_0.$$

Using transformations (33), we obtain the system

$$\begin{aligned} v^0 &= \sin\left[x_0 + a\left(\tfrac{C}{2}(v^1)^2 - C\cos v^2 + D\right)\right], \\ v^1 &= x_1 + a\left(C v^1 \sin v^2 + D v^1\right), \\ v^2 &= x_0 + a\left[\frac{C}{2}(v^1)^2 - C\cos v^2 + D\right]. \end{aligned} \tag{35}$$

Thus, in order to find new solutions of equations (29), it is necessary to solve the overdetermined but compatible system

$$\begin{aligned} u_{00} &= \sin\left[x_0 + a\left(\tfrac{C}{2}(u_{01})^2 - C\cos u_{11} + D\right)\right], \\ u_{01} &= x_1 + a\left(C u_{01} \sin u_{11} + D u_{01}\right), \\ u_{11} &= x_0 + a\left[\frac{C}{2}(u_{01})^2 - C\cos u_{11} + D\right], \end{aligned} \tag{36}$$

where $u_{00} \equiv \dfrac{\partial^2 u}{\partial x_0^2}, u_{01} \equiv \dfrac{\partial^2 u}{\partial x_0 \partial x_1}, u_{11} \equiv \dfrac{\partial^2 u}{\partial x_1^2}$.
The maximal invariance group of point transformations of Eq. (29) is the 7-parameter group. The basis elements of the corresponding algebra are

$$\begin{aligned} &P_0 = \partial_{x_0}, \quad P_1 = \partial_{x_1}, \quad P_2 = \partial_u, \\ &D = x_0\partial_{x_0} + x_1\partial_{x_1} + 2u\partial_u, \\ &Q_0 = x_0\partial_u, \quad Q_1 = x_1\partial_u, \quad Q_2 = x_0 x_1 \partial_u \end{aligned} \tag{37}$$

It can be shown that system (36) has no solutions invariant under the operator $\alpha_0 P_0 + \alpha_1 P_1 + \alpha_2 P_2 + dD + \beta_0 Q_0 + \beta_1 Q_1 + \beta_2 Q_2$, where $\alpha_0, \alpha_1, \alpha_2, d, \beta_0, \beta_1, \beta_2$ are arbitrary constants. Therefore, any solution of system (36) is not an invariant solution for Eq.(29).

It seems that the existence of the infinite-dimensional invariance algebra of system (30) allows linearization of this system by means of hodograph transformations

$$x_0 = x_0\left(v^2, v^3\right), \quad x_1 = x_1\left(v^2, v^3\right) \tag{38}$$

In fact, excepting v^1 in (30), we obtain the system

$$\begin{aligned} &F_1 v_1^2 + F_2 v_1^3 = v_1^1, \\ &v_0^3 = v_1^2. \end{aligned} \tag{39}$$

In terms of variables given by (38), this system will be written in the form

$$\begin{aligned} &F_1\left(v^2, v^3\right)\frac{\partial x_0}{\partial v^3} - F_2\left(v^2, v^3\right)\frac{\partial x_0}{\partial v^2} = -\frac{\partial x_1}{\partial v^3}, \\ &\frac{\partial x_1}{\partial v^2} = \frac{\partial x_0}{\partial v^3}. \end{aligned} \tag{40}$$

Integrating the linear system (40), one can obtain the exact solutions of Eq.(29) as well as construct the nonlinear superposition principle for its solutions.

References

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