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Two-Parameter Deformation of the Oscillator Algebra and (p,q)–Analog of Two-Dimensional Conformal Field Theory

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Abstract

The two-parameter deformation of canonical commutation relations is discussed. The self-adjointness property of the (p,q)-deformed position Q and momentum P operators is investigated. The (p,q)-analog of two-dimensional conformal field theory based on the (p,q)-deformation of the $su(1,1)$ subalgebra of the Virasoro algebra is presented.

1. The one-parameter deformation of canonical commutation relations has arisen from study of the dual resonance models [1]. The more general deformation of these relations has been proposed in [2, 3] to construct the Jordan-Schwinger realization of quantum algebras. Although one-parameter deformations have been mostly studied, the multiparameter ones have aroused much interest because they become more flexible when we are dealing with applications to concrete physical models. The two-parametric deformations of the quantum algebras have been studied in [4-9]. By means of the contraction procedure of the two-parameter deformed quantum group $U_{pq}(sl(2))$, the two-parameter deformed canonical commutation relations of the oscillator algebra have been obtained in [8]. The self-adjointness property of position and momentum operators of the (p,q)-deformed oscillator algebra has been investigated in [10].

The (p,q)-deformed oscillator algebra is defined by three generators a, a_+, N satisfying the following (p,q)-deformed canonical commutation relations

$$\begin{aligned} aa_+ - qa_+a &= p^{-N}, \qquad & [N,a] &= -a, \\ aa_+ - p^{-1}a_+a &= q^N, \qquad & [N,a_+] &= a_+. \end{aligned} \tag{1}$$

From the relations (1), one obtains

$$a_+a = [N], \qquad aa_+ = [N+1], \tag{2}$$

where $[x]$ means $[x]_{qp} = (q^x - p^{-x})/(q - p^{-1})$. The two-parameter deformation of the commutation relations (1) is a generalization of the one-parameter deformation. Putting in (1) $(q,p) = (q,1)$, one gets the one-parameter deformation of the canonical commutation relations [1]

$$aa_+ - qa_+a = 1 \tag{3}$$

and choosing $(q,p) = (q,q)$, one has the other one-parameter deformation of these relations [2,3]

$$\begin{aligned} aa_+ - qa_+a &= q^{-N}, \qquad & [N,a] &= -a, \\ aa_+ - q^{-1}a_+a &= q^{N}, \qquad & [N,a_+] &= a_+. \end{aligned} \tag{4}$$

The action of the operators a, a_+ and N

$$a|n\rangle = [n]_{qp}^{1/2}|n-1\rangle, \qquad a_+|n\rangle = [n+1]_{qp}^{1/2}|n+1\rangle, \qquad N|n\rangle = n|n\rangle \tag{5}$$

on the basis vectors $|n\rangle, n = 1,2,...$, of the Hilbert space defines the Fock representation of the commutation relations (1). It is naturally to define the (q,p)-deformed position Q and momentum P operators by the formulas

$$Q|n\rangle = 2^{-1/2}(a_+|n\rangle + a|n\rangle), \qquad P|n\rangle = i2^{-1/2}(a_+|n\rangle - a|n\rangle). \tag{6}$$

Each of operators (6) is defined by the symmetrical Jacobi matrix

$$\begin{pmatrix} b_0 & a_0 & 0 & 0 & \dots \\ a_0 & b_1 & a_1 & 0 & \dots \\ 0 & a_1 & b_2 & a_2 & \dots \\ . & . & . & . & . \end{pmatrix}. \tag{7}$$

If a_k and $b_k, k = 0,1,2,...$ in (7) are bounded, then the operator defined by this matrix is bounded (Theorem 1.2, Chapter VII in ref. [11]). Depending on the values of the parameters q and p, one has

$$\begin{aligned} &\lim_{n\to\infty}[n]_{qp} = \infty,\ q>1,\ p>1, \\ &\lim_{n\to\infty}[n]_{qp} = \infty,\ q<1,\ p<1, \\ &\lim_{n\to\infty}[n]_{qp} = 0,\ q<1,\ p>1, \\ &\lim_{n\to\infty}[n]_{qp} = \infty,\ q>1,\ p<1,\ qp>1, \\ &\lim_{n\to\infty}[n]_{qp} = \infty,\ q>1,\ p<1,\ qp<1. \end{aligned} \tag{8}$$

The operators $a_+ + a, i(a_+ - a)$ are bounded only in the third case of (8) and unbounded otherwise. It can be shown that operators $a_+ + a, i(a_+ - a)$ are simultaneously self-adjoint or not self-adjoint. Let us consider the operator $a_+ + a$. In this case we have in (7) $b_n = 0, a_n = [n]_{qp}^{1/2}, n = 0,1,...$. In accordance with the Theorem 1.5 from Chapter VII, in [11], the operator defined by matrix (7) is self-adjoint if the series

$$\sum_{n=1}^{\infty} 1/a_n \tag{9}$$

composed from the quantities reversed to a_n is divergent. If the series (9) converges and in addition the conditions

$$a_{n-1}a_{n+1} \le a_n^2, n = 1,2,... \tag{10}$$

are fulfilled, then the operator defined by the matrix (7) is not self-adjoint (Theorem 1.5, Chapter VII, [11]). It can be shown with the help of the inequality $a+a^{-1}\geq 2$ that

$$[n-1]_{qp}^{1/2}[n+1]_{qp}^{1/2}\leq [n]_{qp}, n=0,1,2,... \tag{11}$$

for all values q,p. Therefore, the self-adjointness or not self-adjointness of the operator $a_{+}+a$ is defined by the divergence or convergence of the series (9). In our case the series

$$\sum_{n=0}^{\infty} 1/[n]_{pq}^{1/2} \tag{12}$$

is divergent if $q>1, qp>1$ or $p<1, qp<1$ and convergent if $q>1, qp>1$ or $p>1, qp<1$. One can conclude from (11) and (12) that the operator $a_{+}+a$ is self-adjoint only if it is the bounded operator, that is, if $q<1, p>1$ and does not admit a self-adjoint closure for all other values of the parameters q and p. According to Theorem 1,1., Chapter VII in [11], the deficiency indices of this operator are equal to (1,1). This means that deficiency subspaces are one–dimensional. Besides, deficiency subspaces N_z, $\mathrm{Im} z\neq 0$ are defined by the generalized vectors $|z\rangle=\sum_{n=0}^{\infty}P_n(z)|n\rangle$ such that

$$[n]_{qp}^{1/2}P_{n-1}(z)+[n+1]_{pq}^{1/2}P_{n+1}(z)=zP_n(z) \tag{13}$$

with the initial conditions $P_{-1}(z)=0,\ \ P_0(z)=1$. The solution of equation (13) with these initial conditions has the form $P_n(z)=\{[n]_{qp}!\}^{-1/2}H_n^{qp}(z)$, where polynomials $H_n^{qp}(z)$ satisfy the recurrence relation

$$[n]_{qp}H_{n-1}^{qp}(z)+H_{n+1}^{qp}(z)=zH_n^{qp}(z), n=0,1,2,..., \tag{14}$$

$H_0^{qp}(z)=1$. Repeating the reasoning of the paper [12], it can be proved that

$$H_n^{pq}(z)=\sum_{k=1}^{[n/2]}c_k z^{n-2k} \tag{15}$$

where $[n/2]$ means the integral part of the number $n/2$ and

$$c_k=(-1)^k\sum_{m_k=2k-1}^{n-1}[m_k]_{pq}\sum_{m_{k-1}=2k-3}^{m_k-2}[m_{k-1}]_{pq}\sum_{m_{k-2}=2k-3}^{m_{k-1}-2}[m_{k-2}]_{pq}\dots\sum_{m_1=1}^{m_2-2}[m_1]_{pq} \tag{16}$$

In order to construct the representation of the commutation relations (1) in the space of analytic functions, we consider the (q,p)-difference derivative [6]

$$\tilde{D}_{qp}f(z)=(f(qz)-f(p^{-1}z))/(q-p^{-1})z. \tag{17}$$

The operator $\tilde{D}_{qp}$ is a pseudo-differential operator in the space of analytic functions

$$\tilde{D}_{qp}f(z)=\sum_{n=0}^{\infty}\frac{(q-1)^{n+1}-(p^{-1}-1)^{n+1}}{q-p^{-1}}\frac{z^n}{n!}\frac{d^{n+1}}{dz^{n+1}}f(z). \tag{18}$$

The basic properties of the (q,p)-derivative $\tilde{D}_{qp}$ are

$$\begin{aligned}&\tilde{D}_{qp}c=0, \qquad c\in\mathbf{C}\\&\tilde{D}_{qp}(f_1(z)f_2(z))(\tilde{D}_{qp}f_1)(z)f_2(qz)+f_1(p^{-1}z)(\tilde{D}_{qp}f_2)(z).\end{aligned} \tag{19}$$

By analogy to the definition of the q-exponential function $\exp_q(z)$, one defines the (q,p)-exponential function [7]

$$\exp_{qp}(z) = \sum_{n=1}^{\infty} z^n/[n]_{qp}!, \qquad [n]_{qp}! = [n]_{qp} \dots [1]_{qp}. \tag{20}$$

From (17) and (20), one obtains

$$\tilde{D}_{qp} \exp_{qp}(\mu z) = \mu \exp_{qp}(\mu z). \tag{21}$$

We denote by F the space of the analytic functions in the domain $\{z \in C, |z|^2 < z_0\}$,where $-z_0 (z_0 > 0)$ is the largest root of the function (20). In this space, the operators

$$a_+ f(z) = zf(z),\ af(z) = D_{qp} f(z),\ Nf(z) = zdf(z)/dz \tag{22}$$

define a representation of the commutation relations (1). To construct the scalar product in the space F, we define a Jackson (q,p)-integral. It is defined by

$$I_{qp} = \sum_{k=1}^{\infty} (q^{-k}p^{-k} - q^{-k-1}p^{-k-1}) af(q^{-k-1}p^{-k}a), \quad |qp| > 1 \tag{23}$$

and

$$I_{qp} = \sum_{k=1}^{\infty} (q^k p^k - q^{k+1}p^{k+1}) af(q^k p^{k+1} a), \quad |qp\ | < 1. \tag{24}$$

A simple calculation yields

$$\int_0^a f_1(p^{-1}z) \tilde{D}_{qp} f_2(z) d_{qp} z = [f_1(z) f_2(z)]|_0^a - \int_0^a f_2(qz) \tilde{D}_{qp} f_1(z) d_{qp} z, \tag{25}$$

$$\int_0^a f(sz) d_{qp} z = \frac{1}{s} \int_0^{sa} f(z) d_{qp} z. \tag{26}$$

Making use of (24) and (25), it is easy to find

$$\int_0^a z^m \exp_{qp}(-z)\, d_{qp} z = (q^{-1}p)^{m(m+1)/2} [m]_{qp}! \tag{27}$$

In the space F we can define the scalar product

$$(f,g) = \frac{1}{2\pi} \int_0^{z_0} \left(\int_0^{2\pi} \bar{f}(z) g(z) d\theta \right) \exp_{qp}(-|z|^2) d_{qp} |z|^2, \tag{28}$$

where $z = |z| \exp i\theta$. The set of the functions

$$u_n(z) = (qp^{-1})^{n(n+1)/4} z^n / \{[n]_{qp}!\}^{1/2}, n = 1, 2, \dots \tag{29}$$

of the space F form an orthonormal system with respect to the scalar product (28).

Now we investigate the Hermitian conjugation of the operators a and a_+ with respect to (28). Expanding the functions $f(z)$ and $g(z)$ in the orthonormal system (29), one has

$$\begin{aligned}(zf,g) &= \sum_{n=0}^{\infty} \bar{f}_n g_{n+1}[n+1]_{qp}^{1/2}(q^{-1}p)^{(n+1)/2},\\ (f,\tilde{D}_{qp}g) &= \sum_{n=0}^{\infty} \bar{f}_n g_{n+1}[n+1]_{qp}^{1/2}(qp^{-1})^{(n+1)/2}.\end{aligned} \tag{30}$$

Comparing the right–hand sides of equations (30), we see that the operators z and $\tilde{D}_{qp}$ are Hermitian conjugate

$$(zf,g) = (f,\tilde{D}_{qp}g) \tag{31}$$

with respect to the scalar product (28) if $q^2 = p^2$. If $q = p$, we have the operator (17) and the representation of the commutation relations (4). If $q = -p$, one obtains the difference operator

$$\tilde{D}_{q,-q}f(z) = \frac{f(qz) - f(-q^{-1}z)}{(q+q^{-1})z} \tag{32}$$

and the representation of the $(p,q) = (q,-q)$ one-parameter deformed canonical commutation relations.
2. The generalizations of conformal field theory based on deformations of the symmetry algebra have been studied in [14-19]. The space of states of conformal field theory is an inner product space carrying the representation of the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}n(n+1)\delta_{m+n,0}, \quad n = 0, \pm 1, \pm 2, \dots \tag{33}$$

The properties of correlation functions of the theory are determined by the Ward identities for the subalgebra $su(1,1)$ of the Virasoro algebra (33)

$$[E_0, E_{+1}] = E_{+1}, \quad [E_0, E_{-1}] = -E_{-1}, \quad [E_{-1}, E_{+1}] = -2E_0, \tag{34}$$

where $E_0 = -L_0, E_{-1} = L_{-1}, E_{+1} = L_{+1}$. The universal enveloping algebra $U(su(1,1))$ of the Lie algebra (34) admits the Hopf algebra structure. In particular, the operation of the comultiplication is defined as

$$\Delta(E_n) = E_n \otimes 1 + 1 \otimes E_n, \quad n = 0, \pm 1. \tag{35}$$

The homomorphism $\Delta : su(1,1) \to su(1,1) \otimes su(1,1)$ can be extended to the one $\Delta^N : su(1,1) \to \otimes_{i=1}^{N} su(1,1)$ by the formula

$$\Delta^N(E_n) = (\Delta \otimes \mathrm{id} \otimes \ldots \otimes \mathrm{id}) \ldots (\Delta \otimes \mathrm{id})\Delta(E_n), \quad n = 0, \pm 1. \tag{36}$$

There exist various deformations of the universal enveloping algebra $U(su(1,1))$ preserving the Hopf algebra structure. As such it is the (p,q)-deformation $U_{pq}(su(1,1))$ of this algebra. The generators K_{+1}, K_{-1}, K_0 of the quantum algebra $U_{pq}(su(1,1))$ satisfy the following commutation relations [7]

$$\begin{aligned}&[K_0, K_{+1}] = K_{+1}, \quad [K_0, K_{-1}] = -K_{-1},\\ &[K_{-1}, K_{+1}]_{pq} = K_{-1}K_{+1} - qp^{-1}K_{+1}K_{-1} = [2K_0]_{pq},\end{aligned} \tag{37}$$

where $[a]_{pq} = (q^a - p^{-a})/(q - p^{-1})$ and p, q are complex parameters. The algebra (37) admits the Hopf algebra structure. In particular, the operation of the comultiplication is given by the formula

$$\Delta(K_{\pm 1}) = q^{K_0} \otimes K_{\pm 1} + K_{\pm 1} \otimes p^{-K_0}, \Delta(K_0) = K_0 \otimes 1 + 1 \otimes K_0. \tag{38}$$

The representation of the commutation relations (37) on the space F of functions $f(z)$ is defined as

$$\begin{aligned} &K_{+1} f(z) = z(q^{2h} f(zq) - p^{-2h} f(zp^{-1})/(q - p^{-1}), \\ &K_{-1} f(z) = (1/z)(f(zq) - f(zp^{-1})/(q - p^{-1}), \\ &\mathcal{K}_q f(z) \stackrel{\text{def}}{=} q^{K_0} f(z) = q^h f(qz), \end{aligned} \tag{39}$$

where h is a conformal dimension of quasi-conformal field. A quasi-primary field $\phi_h(z)$ with the conformal dimension h is transformed under $U_{pq}(su(1,1)$ as

$$\begin{aligned} &[\hat{K}_n, \phi_h] = \{z^n [(n+1)h]_{pq} \phi_h(zq) + p^{-(n+1)h} z^{n+1} (D_{pq}\phi_h)(z)\} \hat{\mathcal{K}}_p^{-1}, \quad n = \pm 1, \\ &\hat{\mathcal{K}}_q \phi_h(z) \hat{\mathcal{K}}_p^{-1} = q^h \phi_h(zq). \end{aligned} \tag{40}$$

The commutator on the left–hand side of the first equality in (40) is defined as

$$[A, \phi_h(z)] = A\phi_h(z) - \hat{\mathcal{K}}_q \phi_h(z) \hat{\mathcal{K}}_q^{-1} A \tag{41}$$

with $A \in U_{pq}(su(1,1))$. The formulae (40) and (41) at $p = 1$ coincide with the formulae (7) and (8) of [15] and at $p = q = 1$ we obtain

$$[\hat{K}_n, \phi_h(z)] = z^n [z\partial_z + h(n+1)] \phi_h(z), \quad n = 0, \pm 1, \tag{42}$$

that is the transformation law of primary fields of conformal field theory. The $U_{pq}(su(1,1)$ invariant vacuum $|0\rangle$, $\hat{K}_{\pm 1}|0\rangle = 0$, $\hat{\mathcal{K}}_q \| 0\rangle = |0\rangle$, and quasi-primary fields $\phi_{h_1}(z)$, $\phi_{h_2}(z)$, $\ldots$, $\phi_{h_N}(z)$ of the conformal weights $h_1, h_2, \ldots, h_N$, respectively, define the correlation functions

$$\langle \phi_1(z_1) \ldots \phi_N(z_N) \rangle_{pq} = \langle 0 | \phi_{h_1}(z_1) \ldots \phi_{h_N}(z_N) | 0 \rangle_{pq}. \tag{43}$$

Using the commutation relations (40), (41) and the $U_{pq}(su(1,1))$ invariance of the vacuum, we obtain the equations which provide the $U_{pq}(su(1,1))$ invariance of the correlation functions

$$\begin{aligned} &0 = \langle \hat{K}_n \phi_1(z_1) \ldots \phi_N(z_N) \rangle = \textstyle\sum_{j=1}^N q^{h_1 + h_2 + \ldots + h_{j-1}} p^{-h_{j+1} - \ldots - h_N} \\ &\times \langle \phi_1(qz_1) \ldots \phi_{j-1}(qz_{j-1}) \hat{\phi}_i(z_i) \phi_{j+1}(p^{-1} z_{j+1} \ldots \phi_N(p^{-1} z_N) \rangle_{pq}, \end{aligned} \tag{44}$$

$$\langle \hat{\mathcal{K}}_q \phi_1(z_1) \ldots \phi_N(z_N) \rangle_{pq} = q^{h_1 + h_2 + \ldots + h_N} \langle \phi_1(qz_1) \ldots \phi_N(qz_N) \rangle_{pq}, \tag{45}$$

where $\hat{\phi}(z) = \{[(n+1)h]_{pq} z^n \phi(qz) + z^{n+1} D_{pq} \phi(z)\} \hat{\mathcal{K}}_p^{-1}$, $\quad n = \pm 1$. With the help of (38) and (39), the equation (44) can be rewritten as

$$\Delta^N(K_{\pm 1}) \langle \phi_1(z_1) \phi_2(z_2) \ldots \phi_N(z_N) \rangle_{pq} = 0. \tag{46}$$

The equations (45), (46) define the (p,q)-deformation of the $su(1,1)$ Ward identities of conformal–invariant field theory. The identities (45),(46) with the help of (38) can be rewritten as

$$\begin{aligned}&\Delta(K_{\pm 1})\langle\phi_i(z_1)\phi_j(z_2)\rangle_{pq} = (q^{K_0}\otimes K_{\pm 1} + K_{\pm 1}\otimes p^{-K_0})\langle\phi_i(z_1)\phi_j(z_2)\rangle_{pq} = 0,\\ &\Delta(\mathcal{K}_p)\langle\phi_i(z_1)\phi_j(z_2)\rangle_{pq} = \langle\phi_i(z_1)\phi_j(z_2)\rangle_{pq}.\end{aligned} \tag{47}$$

From (47) we obtain the following set of difference equations

$$\begin{aligned}&(p^{-h_2}/z_1)\langle\phi_i(p^{-1}z_1)\phi_j(p^{-1}z_2)\rangle_{pq} - ((p^{-h_2}/z_1) - (q^{h_1}/z_2)\langle\phi_i(qz_1)\phi_j(p^{-1}z_2)\rangle_{pq}\\ &-(q^{h_1}/z_2)\langle\phi_i(qz_1)\phi_2(qz_2)\rangle_{pq} = 0,\\ &p^{-2h_1-h_2}z_1\langle\phi_i(p^{-1}z_1)\phi_j(p^{-1}z_2)\rangle_{pq} - (q^{2h_1}p^{-h_2}z_1 - q^{h_1}p^{-2h_2}z_2)\\ &\times\langle\phi_i(qz_1)\phi_j(p^{-1}z_2)\rangle_{pq} - q^{h_1+2h_2}z_2\langle\phi_i(qz_1)\phi_j(qz_2)\rangle_{pq} = 0,\\ &q^{h_i+h_j}\langle\phi_i(qz_1)\phi_j(qz_2)\rangle_{pq} = \langle\phi_i(z_1)\phi_j(z_2)\rangle_{pq}.\end{aligned} \tag{48}$$

The set of equations (48) is consistent and admits a solution if and only if the two conformal weights h_1 and h_2 are equal: $h_1 = h_2 = h$. A solution of the set of equations (48) can be obtained by the following ansatz

$$\langle\phi_1(z_1)\phi_2(z_2)\rangle_{pq} = C(p,q)z^{-a}{}_1\phi_0^{pq}(a;(pq)^{\alpha}z_2/z_1), \tag{49}$$

where the function ${}_n\phi_{n-1}^{pq}(a_1,\ldots,a_n;b_1,\ldots,b_{n-1};z)$ is a (p,q)-hypergeometric function (17) of [13]. The solution (49) of the set of equations (48) can be written as

$$\langle\phi_1(z_1)\phi_2(z_2)\rangle_{pq} = C(p,q)z_1^{-2h}\phi_0^{pq}(2h;(pq)^{1-h}z_2/z_1) \tag{50}$$

In [17-18], the solution (49) has been represented in some other form. The (p,q)-deformed Ward identities (44) for the three-point correlation function $\langle\phi_i(z_1)\phi_j(z_2)\phi_k(z_3)\rangle_{pq}$ can be rewritten as

$$\begin{aligned}&(K_{+1}\otimes p^{-K_0}\otimes p^{-K_0} + q^{K_0}\otimes K_{+1}\otimes p^{-K_0} + q^{K_0}\otimes q^{K_0}\otimes K_{+1})\\ &\langle\phi_i(z_1)\phi_(z_2)\phi_k(z_3)\rangle_{pq} = 0,\\ &(K_{-1}\otimes p^{-K_0}\otimes p^{-K_0} + q^{K_0}\otimes K_{-1}\otimes p^{-K_0} + q^{K_0}\otimes q^{K_0}\otimes K_{-1})\\ &\times\langle\phi_i(z_1)\phi_j(z_2)\phi_k(z_3)\rangle_{pq} = 0,\\ &q^{h_1+h_2+h_3}\langle\phi_i(qz_1)\phi_j(qz_2)\phi_k(qz_3)\rangle_{pq} = \langle\phi_i(z_1)\phi_j(z_2)\phi_k(z_3)\rangle_{pq}.\end{aligned} \tag{51}$$

The set of equations (51) reduces to the following set of difference equations

$$\begin{aligned}&p^{-2h_1-h_2-h_3}z_1\langle\phi_i(p^{-1}z_1)\phi_j(p^{-1}z_2)\phi_k(p^{-1}z_3)\rangle_{pq}\\ &-(p^{-h_2-h_3}q^{2h_1}z_1 - p^{-2h_2-h_3}q^{h_1}z_2)\langle\phi_i(qz_1)\phi_j(p^{-1}z_2)\phi_k(p^{-1}z_3)\rangle_{pq}\\ &-(p^{-h_3}q^{h_1+2h_2}z_2 - p^{-2h_3}q^{h_1+h_2}z_3)\langle\phi_i(qz_1)\phi_j(qz_2)\phi_k(p^{-1}z_3)\rangle_{pq}\\ &-q^{h_1+h_2+2h_3}z_3\langle\phi_i(qz_1)\phi_j(qz_2)\phi_k(qz_3)\rangle_{pq} = 0,\\ &(1/z_1)p^{-h_2-h_3}\langle\phi_i(p^{-1}z_1)\phi_j(p^{-1}z_2)\phi_k(p^{-1}z_3)\rangle_{pq}\\ &-((1/z_1)p^{-h_2-h_3} - (1/z_2)p^{-h_3}q^{h_1})\langle\phi_i(qz_1)\phi_j(p^{-1}z_2)\phi_k(p^{-1}z_3)\rangle_{pq}\\ &-(1/z_2)p^{-h_3}q^{h_1} - (1/z_3)q^{h_1+h_2})\langle\phi_i(qz_1)\phi_j(qz_2)\phi_k(p^{-1}z_3)\rangle_{pq}\\ &-(1/z_3)q^{h_1+h_3}\langle\phi_i(qz_1)\phi_j(qz_2)\phi_k(qz_3)\rangle_{pq} = 0,\\ &q^{h_1+h_2+h_3}\langle\phi_i(qz_1)\phi_j(qz_2)\phi_k(qz_3)\rangle_{pq} = \langle\phi_i(z_1)\phi_j(z_2)\phi_k(z_3)\rangle_{pq}.\end{aligned} \tag{52}$$

This set of equations is consistent and completely defines the three-point correlation function of the quasi-primary fields

$$
\begin{aligned}
&\langle \phi_i(z_1)\phi_j(z_2)\phi_k(z_3)\rangle_{pq} = C_{ijk}(p,q) z_1^{-\gamma_{12}^3-\gamma_{31}^2} z_2^{-\gamma_{23}^1} \\
&\times {}_1\phi_0^{pq}(\gamma_{12}^3;(pq)^{1-h_1}z_2/z_1)\ {}_1\phi_0^{pq}(\gamma_{23}^1;(pq)^{1-h_2}z_3/z_2)\ {}_1\phi_0^{pq}(\gamma_{31}^2;(pq)^{1-h_1+h_2}z_3/z_1)
\end{aligned} \tag{53}
$$

with $\gamma_{ij}^k = h_i + h_j - h_k$. The three-point correlation function (53) in the limiting cases $p = 1$ and $p = q \to q^{-1}$ coincides, respectively, with the one of [14] and [15].

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