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On the Spectral Theory of Operator Pencils in a Hilbert Space

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Abstract

Consider the operator pencil $L_\lambda = A - \lambda B - \lambda^2 C$, where A, B, and C are linear, in general unbounded and nonsymmetric, operators densely defined in a Hilbert space H. Sufficient conditions for the existence of the eigenvalues of L_λ are investigated in the case when A, B and C are K-positive and K-symmetric operators in H, and a method to bracket the eigenvalues of L_λ is developed by using a variational characterization of the problem (i) $L_\lambda u = 0$. The method generates a sequence of lower and upper bounds converging to the eigenvalues of L_λ and can be considered an extension of the Temple-Lehman method to quadratic eigenvalue problems (i).

1 Introduction

Let H be a separable complex Hilbert space with the inner product and norm

$$(x, y),\ ||x|| = (x, x)^{1/2}\ , \qquad (x, y \in H) \tag{1}$$

and consider in H the nonlinear eigenvalue problem

$$Ax - \lambda Bx - \lambda^2 Cx = 0 \tag{2}$$

where A and C are K-p.d. operators with $D_C \supseteq D_A$, D_A is dense in H, and B is an operator with $D_B \supseteq D_C$. Recall [1–3] that by the definition of A and C there exists a closable operator K with $D_K \supseteq D_C$ mapping D_A onto a dense subset KD_A of H and positive constants α_1, α_2, β_1, β_2 such that

$$(Ax, Kx) \geq \alpha_1 ||x||^2\ , \quad (x \in D_A) \tag{3}$$

$$||Kx||^2 \leq \alpha_2 (Ax, Kx)\ , \quad (x \in D_A) \tag{4}$$

$$(Cx, Kx) \geq \beta_1 ||x||^2\ , \quad (x \in D_C) \tag{5}$$

$$||Kx||^2 \leq \beta_2 (Cx, Kx)\ , \quad (x \in D_C) \tag{6}$$

The class of K-p.d. operators $\{P\}$ contains, among others, the following families of mappings:

(a) Positive definite operators; in this case K is the identity map or, if P is also self-adjoint, K can be any root of P.
(b) Closeable and densely invertible[1] operators; in this case we let $K = P$.
(c) The operator P of the form $P = -S^{2j+1}$ or $P = S^{2j+2}$ where for some i, $0 \leq i < j$, the operator $S^{2(j+i+1)}$ is positive definite; in this case we let $K = S^{2i+1}$ or $K = S^{2i+2}$, provided that K so defined is closable and KD_P is dense in H. To this class belong, in particular, ordinary differential operators of odd and even order and weakly elliptic partial differential operators of odd and even order which in general are not self-adjoint [2].
(d) A subclass of bounded and unbounded symmetrizable operators investigated by a number of authors [2, 4].

Let $D[A]$ be the set D_A endowed with the new metric

$$(x,y)_A = (Ax, Ky), \qquad ||x||_A^2 = (x,x)_A, \qquad (x, y \in D_A) \tag{7}$$

and denote by H_A the completion of $D[A]$ in the metric (7). Similarly, let $D[C]$ be the set D_C with the metric

$$(x,y)_C = (Cx, Ky), \qquad ||x||_C^2 = (x,x)_C, \qquad (x, y \in D_C) \tag{8}$$

and define H_C to be the completion of $D[C]$ in the metric (8). One can show that the space H_A is contained in H in the sense of uniquely identifying the elements of H_A with certain elements in H and clearly, since C is K-p.d., the above assertion is valid also for the space H_C, i.e., $H_C \subseteq H$. Let $H_1 = H \times H_C$ be the Cartesian product space, with the norm and inner product defined by

$$(u,v)_1 = (x,p) + (y,q)_C$$

$$\left(u = \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } v = \begin{pmatrix} p \\ q \end{pmatrix} \in H \times H_C\right) \tag{9}$$

$$||u||_1 = (u,u)_1^{1/2} = \left(||x||^2 + ||y||_C^2\right)^{1/2}. \tag{10}$$

Clearly, H_1 is a Hilbert space and, since H_C is a subset of H, it follows that $H_1 \subseteq H \times H$ in the sense mentioned above. Now, let $T : D_A \times D[C] \subseteq H_1 \to H_1$ be the operator matrix

$$T = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}, \; T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Ax \\ y \end{pmatrix}, \quad \left(u = \begin{pmatrix} x \\ y \end{pmatrix} \in D_A \times D[C]\right). \tag{11}$$

Similarly, let us define in H_1 the operators

$$S = \begin{pmatrix} B & C \\ I & 0 \end{pmatrix}, \; S : D_B \times D[C] \subseteq H_1 \to H_1, \tag{12}$$

$$\hat{K} = \begin{pmatrix} K & 0 \\ 0 & I \end{pmatrix}, \; \hat{K} : D_K \times D[C] \subseteq H_1 \to H_1. \tag{13}$$

[1] An operator P will be called invertible if it has a bounded inverse, densely invertible if it is invertible and its range R_P is dense in H, and continuously invertible if it is densely invertible and $R_P = H$.

Observe that the quadratic eigenvalue problem (2) is equivalent to the system

$$\begin{aligned} Ax - \lambda Bx - \lambda Cy &= 0 \\ y - \lambda x &= 0 \end{aligned} \tag{14}$$

which, in view of (11) and (12), is equivalent to the linear equation

$$Tu - \lambda Su = 0 \tag{15}$$

in the sense that if x_i is a solution of (2) corresponding to $\lambda = \lambda_i$, then $u_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$ with $y_i = \lambda_i x_i$ is a solution of (15) and, conversely, if u_i is a solution of (15) corresponding to $\lambda = \lambda_i$, then $y_i = \lambda_i x_i$ and x_i is a solution of (2).

Proposition 1. The operator T defined by (11) is $\hat{K}$-p.d. in the space $H_1 = H \times H_C$; i.e., T satisfies the following conditions:

(a) D_T is dense in H_1.

(b) $D_{\hat{K}} \supseteq D_T$ and $\hat{K}D_T$ is dense in H_1.

(c) $\hat{K}$ is closable in H_1.

(d) There exist positive constants γ_1, γ_2 such that

$$(Tu, \hat{K}u)_1 \geq \gamma_1 ||u||_1^2 \ , \quad (u \in D_T), \tag{16}$$

$$||\hat{K}u||_1^2 \leq \gamma_2 (Tu, \hat{K}u)_1 \ , \quad (u \in D_T). \tag{17}$$

Proof. (a) Let $u = \begin{pmatrix} x \\ y \end{pmatrix}$ be an arbitrary element in $H_1 = H \times H_C$. Since D_A is dense in H, there exists a sequence $\{x_n\} \subset D_A$ which converges to x in the H-metric. Similarly, since D_C is dense in H_C, there exists a sequence $\{y_n\} \subset D_C$ which converges to y in the H_C-metric. Hence, if we define a sequence in $D_T = D_A \times D[C]$ by $u_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$, then $\lim_{n\to\infty} ||u_n - u||_1^2 = \lim_{n\to\infty} (||x_n - x||^2 + ||y_n - y||_C^2) = 0$.

(b) By definition, $\hat{K}D_T = KD_A \times D[C]$ where KD_A is dense in H. Hence, using a similar argument as in part (a), one can show that $\hat{K}D_T$ is dense in H_1. Moreover, since $D_K \supseteq D_A$, it follows that

$$D_{\hat{K}} = D_K \times D[C] \supseteq D_A \times D[C] = D_T. \tag{18}$$

(c) Let $u_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$ be a sequence in $D_{\hat{K}}$, and $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ an element in H_1 such that the following conditions hold:

$$\lim_{n\to\infty} ||u_n||_1 = 0, \tag{19}$$

$$\lim_{n\to\infty} ||\hat{K}u_n - f||_1 = 0. \tag{20}$$

From (20) we obtain $\lim_{n\to\infty} \sqrt{||Kx_n - f_1||^2 + ||y_n - f_2||_c^2} = 0$ which implies that

$$Kx_n \to f_1 \in H, \tag{21}$$

$$y_n \to f_2 \in H_C, \tag{22}$$

On the other hand, from (19) we deduce that

$$x_n \to 0 \text{ in the } H\text{-norm, and} \tag{23}$$

$$y_n \to 0 \text{ in the } H_C\text{-norm.} \tag{24}$$

In view of (21) and (23) it follows that $f_1 = 0$, since K is closable in H. Moreover, from (22) and (24) it follows that $f_2 = 0$. Hence, $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = 0$, and $\hat{K}$ is closable in H_1.

(d) If $u = \begin{pmatrix} x \\ y \end{pmatrix} \in D_T$, then $(Tu, \hat{K}u)_1 = (Ax, Kx) + (y, y)_C$ and, in view of (3), we obtain the inequality

$$(Tu, \hat{K}u)_1 \geq \alpha_1 ||x||^2 + ||y||_C^2. \tag{25}$$

Let $\gamma_1 = \min\{\alpha_1, 1\}$; then from (25) and (10) it follows that

$$(Tu, \hat{K}u)_1 \geq \gamma_1 ||u||_1^2 \ , \quad (u \in D_T). \tag{26}$$

Since $||\hat{K}u||_1^2 = (Kx, Kx) + (y, y)_C$, it follows from (4) and (11) that

$$||\hat{K}u||_1^2 \leq \alpha_2 (Ax, Kx) + (y, y)_C \leq \gamma_2 (Tu, \hat{K}u)_1 \ , \quad (u \in D_T), \tag{27}$$

where $\gamma_2 = max\{\alpha_2, 1\}$. ■

Let $u = \begin{pmatrix} x \\ y \end{pmatrix}$ and $v = \begin{pmatrix} p \\ q \end{pmatrix}$ be elements of the space $D_T = D_A \times D[C] \subseteq H \times H_C = H_1$ and let us introduce in D_T a new norm and inner product

$$(u, v)_2 = (Tu, \hat{K}v)_1 = (x, p)_A + (y, q)_C \tag{28}$$

$$||u||_2 = \sqrt{||x||_A^2 + ||y||_C^2}. \tag{29}$$

Define by $D[T]$ the linear set D_T endowed with the metric $||\cdot||_2^2 = (\cdot, \cdot)_2$, and observe that

$$D[T] = D[A] \times D[C]. \tag{30}$$

In view of (16), (17) and the fact that T is K-p.d. in H_1, we have the inequalities

$$||u||_2 \geq \sqrt{\gamma_1} ||u||_1 \ , \quad (u \in D_T) \tag{31}$$

$$||\hat{K}u||_1 \leq \sqrt{\gamma_2} ||u||_2 \ , \quad (u \in D_T) \tag{32}$$

Clearly, D_T satisfies all the properties of a Hilbert space, with the possible exception of completeness. Let us denote by H_2 the completion of $D[T]$ in the metric (29).

Proposition 2.

(a) $H_2 = H_A \times H_C$.

(b) H_2 is contained in H_1 in the sense of identifying uniquely the elements from H_2 with certain elements in H_1.

(c) $\hat{K}$ can be extended to a bounded operator $\hat{K}_0$ mapping all of H_2 to H_1 such that $\hat{K} \subset \hat{K}_0 \subset \overline{\hat{K}}$, where $\overline{\hat{K}}$ denotes the closure of $\hat{K}$ in H_1.

(d) T has a unique closed $\hat{K}_0$-p.d. extension T_0 such that $T_0 \supseteq T$, T_0 has a bounded inverse T_0^{-1} defined on all of $H_1 = R_{T_0}$, and the inequalities (31) and (32) remain valid in H_2 in the form

$$||u||_2 \geq \sqrt{\gamma_1}||u||_1 \ , \quad (u \in H_2) \tag{33}$$

$$||\hat{K}_0 u||_1 \leq \sqrt{\gamma_2}||u||_2 \ , \quad (u \in H_2). \tag{34}$$

Proof. The proof of part (a) follows from (30) and the fact that H_A and H_C are the completions of the spaces $D[A]$ and $D[C]$ in the norms $|| \cdot ||_A$ and $|| \cdot ||_C$, respectively. By Proposition 1, the operator T is $\hat{K}$-p.d. in H_1. Hence, the proof of parts (b), (c), and (d) can be derived from Lemma 1.2 of Petryshyn [3], provided the spaces H_2, H_1 and the operator $\hat{K}$ in Proposition 2 are identified with H_0, H, and K, in Lemma 1.2, respectively. ■

In the sequel we shall assume, when necessary, that the operators $\hat{K}$ and T have already been extended and the notation T_0 and $\hat{K}_0$ will not be used. Note that in applications it is often not necessary to extend the operators T and $\hat{K}$.

2 The equaivalent linear problem $Tu - \lambda Su = 0$

Definition 1. The quadratic eigenvalue problem

$$Ax - \lambda Bx - \lambda^2 Cx = 0, \tag{35}$$

where A and C are K-p.d. with $D_A \subseteq D_C \subseteq D_B$ and B is K-symmetrric on D_C, i.e.,

$$(Bx, Ky) = (Kx, By) \ , \quad (x, y \in D_C) \tag{36}$$

will be called K-real.

Proposition 3. If the quadratic eigenvalue problem (35) is K-real in H, then the equivalent linear problem

$$Tu - \lambda Su = 0 \tag{37}$$

defined by (11)-(13) is $\hat{K}$-real in $H_1 = H \times H_C$, i.e. T is $\hat{K}$-p.d. and S is $\hat{K}$-symmetric on D_T.

Proof. In view of Proposition 1, only the $\hat{K}$-symmetry of S needs to be verified. To this end let $u = \begin{pmatrix} x \\ y \end{pmatrix}$ and $v = \begin{pmatrix} p \\ q \end{pmatrix}$ be elements in $D_T \subseteq H_1$ and note that

$$(Su, \hat{K}v)_1 = (Bx + Cy, Kp) + (x, q)_C = (Bx, Kp) + (Cy, Kp) + (Cx, Kq) \tag{38}$$

Since by definition the operators B and C are K-symmetric on $D_A \subseteq H$, the above equation yields the identity

$$(Su, \hat{K}v)_1 = (Kx, Bp + Cq) + (Cy, Kp) = (\hat{K}u, Sv)_1 \ , \quad (u, v \in D_T) \tag{39}$$

which proves the $\hat{K}$-symmetry of S on D_T. ■

Let us assume that the eigenvalue problem (35) is K-real, which implies that problem (37) is $\hat{K}$-real. A value of the complex parameter λ for which (37) has a nontrivial solution $u \in D_T$ will be called an eigenvalue of (37), and u its corresponding eigenfunction. The set of all eigenvalues of (37) will be denoted by $p\sigma(37)$ and called the point spectrum of (37). By the multiplicity of λ we shall mean the number of linearly independent eigenfunctions which correspond to λ. Since T, S are $\hat{K}$-symmetric, it follows [5] that the eigenvalues of (37) are real, and the eigenfunctions u_1, u_2 corresponding to distinct eigenvalues λ_1, λ_2 are orthogonal in the sense that $(Tu_1, \hat{K}u_2)_1 = 0$. Since the space H_1 is separable, it follows that the point spectrum of (37) is countable.

Suppose the operators K and $L_\lambda \equiv A - \lambda B - \lambda^2 C$ are closed with $D_K = D_C$, and that $L_\lambda : D_A \subseteq H \to H$ is a bijection for all λ, except possibly for a discrete set of eigenvalues of the problem $(A - \lambda B - \lambda^2 C)x = 0$. Under the above assumptions, it is not difficult to show that the equivalent linear problem $Tu - \lambda Su = 0$ in $H_1 = H \times H_C$ satisfies the following conditions:

(α): The operator $G_\lambda = T - \lambda S : D_T \subseteq H_1 \to H_1$ is continuously invertible for all $\lambda \notin p\sigma$ (37).

(β): The spectrum $\sigma(N)$ of the operator $N = T^{-1}S : D[T] \subseteq H_2 \to D[T]$, contains only eigenvalues of finite multiplicity with zero as its sole possible limit point.

Let $p\sigma(L_\lambda) = \{\lambda_i : i = 1, 2, ..\}$ denote the point spectrum of the operator L_λ, with the eigenvalues ordered according to increasing magnitude and repeated as many times as their multiplicity indicates. Let $\{x_i : i = 1, 2, \ldots\}$ be the set of corresponding eigenfunctions, normalized in the sense that $||x_i||_A^2 + \lambda_i^2 ||x_i||_C^2 = 1$. Then, using certain results from the theory of linear K-real eigenproblem $Tu - \lambda Su = 0$, we may derive the following theorems, which extend the corresponding results [6–8] obtained for the case when C is the identity operator and A, B are self-adjoint, positive definite, or compact operators. (Related results, under different assumptions on the operators A,B,C, have been obtained by other authors) [9–15].

Theorem 1. *Assume that the eigenproblem (35) is K-real, that $L_\lambda : D_A \to H$ is a bijection for all $\lambda \notin p\sigma(L_\lambda)$ and that the operators L_λ and K are closed with $D_K = D_C$. Then the eigenvalues and eigenfunctions of problem (35) have the variational characterization*

$$\frac{1}{|\lambda_n|} = \sup_{(x,y)^T \in D_A \times D_C} \{|E(x, y)| \,:\, (x, x_i)_A + \lambda_i (y, x_i)_C = 0,\ 1 \le i \le n-1\} = E(x_n, \lambda_n x_n), \tag{40}$$

where $E(x, y) = \dfrac{(Bx, Kx) + 2Re(Cx, Ky)}{(Ax, Kx) + (Cy, Ky)}$.
Moreover, the eigenvalues found by this variational process exhaust entirely the set $p\sigma(L_\lambda)$.

Proof. By hypothesis the linearized eigenproblem (37) is $\hat{K}$-real and satisfies conditions (α) and (β). It follows from the theory of linear K-real eigenproblems [3, 5] that the eigenpairs (λ_i, u_i) of problem (37), normalized in the sense $||u_i||_2 = 1$, satisfy the variational principle

$$\frac{1}{|\lambda_n|} = \sup_{u \in D_T} \left\{ \frac{|(Su, \hat{K}u)_1|}{(Tu, \hat{K}u)_1} \,:\, (Tu, \hat{K}u_i)_1 = 0,\ 1 \le i \le n-1 \right\} =$$

$$|(Su_n, \hat{K}u_n)_1| \;/\; (Tu, \hat{K}u_n)_1 \tag{40a}$$

and the eigenvalues determined by (40a) exhaust entirely the set $p\sigma(37)$. Thus, the validity of the last assertion of Theorem 1 follows from the fact that $p\sigma(37) = p\sigma(L_\lambda)$. If we let $u = (x, y)^T$, $u \in D(T) = D_A \times D_C$, then expanding the inner products in (40a) and using the K-symmetry property of the operator C, we obtain the expressions

$$(Su, \hat{K}u)_1 = (Bx, Kx) + (Cy, Kx) + (Cx, Ky) = (Bx, Kx) + 2Re(Cx, Ky)$$

$$(Tu, \hat{K}u)_1 = (Ax, Kx) + (Cy, Ky)$$

$$||u_i||_2^2 = ||x_i||_A^2 + \lambda_i^2||x_i||_C^2 \quad (Tu, \hat{K}u_i)_1 = (x, x_i)_A + \lambda_i(y, x_i)_C.$$

Substituting the above into (40a) yields the variational formula (40). ■

Lemma 1. *Assume the hypothesis of Theorem 1.*

(a) Suppose S and S^+ are $\hat{K}$-symmetric operators, T is $\hat{K}$-p.d., and

$$|(S^+u, \hat{K}u)| \geq |(Su, \hat{K}u)|$$

for $u \in D_T$. Then the eigenvalues λ_i^+ and λ_i of the corresponding eigenproblems $Tu - \lambda^+S^+u = 0$ and $Tu - \lambda Su = 0$ satisfy the inequality $|\lambda_i^+| \leq |\lambda_i|$, i=i,2,...

(b) Suppose that T and T^* are $\hat{K}$-p.d. operators with $D_T = D_T^*$, S is K-symmetric on D_T, and

$$(T^*u, \hat{K}u) \geq (Tu, \hat{K}u)$$

for $u \in D_T$. Then the eigenvalues λ_i^* and λ_i of the corresponding eigenproblems $Tu - \lambda Su = 0$ and $T^*u - \lambda^*Su = 0$ satisfy the inequality $|\lambda_i^*| \geq |\lambda_i|$, i=1,2,...

Proof. The proof of parts (a) and (b) is a direct consequence of the variational principle (40) in Theorem 1.

Theorem 2. *Assume the hypothesis of Theorem 1 and let $\{u_i : 1, 2, \ldots\}$ be the set of eigenfunctions, orthonormal in H_2, of the $\hat{K}$-real eigenproblem (37). If $u \in D_T$, then $T^{-1}Su$ has the expansion*

$$T^{-1}Su = \sum_{i=1}^{\infty}(Su, \hat{K}u_i)_1 u_i \tag{41}$$

which converges in the H_1 and H_2-norm.

Proof. The result follows directly from the corresponding eigenfunction expansion theorem [3, 5] for linear K-real eigenvalue problems.

3 Iterative method

Let $f_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ be an element in D_T such that $f_0 \notin N(S)$ (the null space of S), and denote by $f_k = \begin{pmatrix} x_k \\ y_k \end{pmatrix}$ the iterant at the k-th step of our process; then the succeeding iterant f_{k+1} is obtained by solving the equation $Tf_{k+1} = Sf_k$, i.e.,

$$\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} B & C \\ I & 0 \end{pmatrix}\begin{pmatrix} x_k \\ y_k \end{pmatrix}, \quad (k \geq 0). \tag{42}$$

Now, let us determine the constants

$$a_k = (Sf_{k-i}, \hat{K}f_i)_1 = (Bx_{k-i}, Kx_i) + (Cy_{k-i}, Kx_i) + (Cx_{k-i}, Ky_i)$$
$$(0 \le i \le k,\ k = 1, 2, \ldots). \tag{43}$$

Note that the values of $(Sf_{k-i}, \hat{K}f_i)_1$ depend on k but not on i, since from the $\hat{K}$-symmetry of S and T it follows that $(Sf_k, \hat{K}f_0)_1 = (Sf_{k-1}, \hat{K}f_1)_1 = \cdots = (Sf_0, \hat{K}f_k)_1$. Also, note that the elements of the sequence $\{f_k\}$ cannot vanish, since $f_0 \notin N(S)$ implies that $f_n \notin N(S)$ for $n \ge 0$. Indeed, if $Sf_n \ne 0$ for $n < k$, then $f_k = T^{-1}Sf_{k-1} \ne 0$, and from the identity $(Sf_k, \hat{K}f_{k-1})_1 = (\hat{K}f_k, Sf_{k-1})_1 = (Tf_k, \hat{K}f_k)_1 > 0$ it follows that $Sf_k \ne 0$. Thus, by induction, it follows that $f_n \notin N(S)$ for all $n \ge 0$.

Let H_2^i be the space spanned by the eigenfunction u_i and denote by $(H_2^i)^\perp$ the orthogonal complement of H_2^i in H_2.

Proposition 4. Let $c_i = (f_0, u_i)_2, i = 1, 2, \ldots$ be the Fourier coefficients of f_0 with respect to the orthonormal set of eigenfunctions $\{u_i\}$ in H_2. Then,

(a) f_k may be represented by the following series, converging in the H_1 and H_2 metrics:

$$f_k = \sum_{i=1}^{\infty} c_i \lambda_i^{-k} u_i\ , \quad (k = 0, 1, \ldots) \tag{44}$$

(b) the constants a_k, determined by (43), are of the form

$$a_k = \sum_{i=1}^{\infty} |c_i|^2 \lambda_i^{-(k+1)}\ , \quad (k = 0, 1, \ldots) \tag{45}$$

Proof. (a) Applying Theorem 2 we may express f_k in the form

$$f_k = T^{-1}Sf_{k-1} = \sum_{i=1}^{\infty} (Sf_{k-1}, \hat{K}u_i)_1 u_i\ , \quad (k = 1, 2, \ldots) \tag{46}$$

where the series converges in the H_1 and H_2 metrics. Now, let us show that the following identity is valid

$$(Sf_{k-1}, \hat{K}u_i)_1 = c_i \lambda_i^{-k}\ , \quad (k = 1, 2, \ldots) \tag{47}$$

For $k = 1$ using the $\hat{K}$-symmetry of S and T, we obtain

$$(Sf_0, \hat{K}u_i)_1 = (\hat{K}f_o, Su_i)_1 = \lambda_i^{-1}(\hat{K}f_0, Tu_i)_1 = \lambda_i^{-1}(Tf_0, \hat{K}u_i)_1 = c_i\lambda_i^{-1}.$$

Suppose (47) is valid for $n < k$, then

$$(Sf_k, \hat{K}u_i)_1 = \lambda_i^{-1}(\hat{K}f_k, Su_i)_1 = \lambda_i^{-1}(Sf_{k-1}, \hat{K}u_i)_1 = c_i\lambda_i^{-(k+1)}.$$

Hence, identity (47) is valid by induction and substituting it into (46) completes the proof of part (a).

(b) Recall that the operator $\hat{K}$, understood in the extended sense, is a continuous mapping from H_2 into H_1 and that the series (44) is convergent in the H_1 and H_2 metrics. Thus, applying the expansion (44) to the last term in the identity

$$a_k = (Sf_k, \hat{K}f_0)_1 = (Tf_{k+1}, \hat{K}f_0)_1 = (\hat{K}f_{k+1}, Tf_0)_1,$$

we obtain

$$a_k = \sum_{i=1}^{\infty} c_i \lambda_i^{-(k+1)} (\hat{K} u_i, T f_0)_1 = \sum_{i=1}^{\infty} |c_i|^2 \lambda_i^{-(k+1)} \ , \quad (k = 0, 1, \ldots). \tag{48}$$

■

Let $w_k = a_{2k-1}/a_{2k+1}$ and note that by applying (45) we may express w_k in the form

$$w_k = \sum_{i=1}^{\infty} |c_i|^2 \lambda_i^{-2k} / \sum_{i=1}^{\infty} |c_i|^2 \lambda_i^{-2(k+1)} \ , \ (k = 1, 2, \ldots). \tag{49}$$

Theorem 3. *Assume the hypothesis of Theorem 1 and suppose that* $|\lambda_r| < |\lambda_{r+1}|$ *for some positive integer* r. *If* f_0 *is chosen from the space*

$$f_0 \in D[T] \ \cap \ [\cap_{i=1}^{r-1} (H_2^i)^{\perp}], \quad f_0 \notin (H_2^r)^{\perp}, \quad r \geq 1, \tag{50}$$

then the following statements are true:
(a) the sequence $\{\sqrt{w_k}\}$ *converges monotonically from above to* $|\lambda_r|$,
(b) $s_k = \lambda_r^{2k} f_{2k}, k = 1, 2, \ldots$ *converges in the* H_2*-metric to an eigenfunction* $c_r u_r \in H_2^r$.

Proof. (a) To show monotonicity of the sequence $\{w_k\}$, let $z_k \in D_T$ be defined by

$$z_k = a_{2k+3} f_k - a_{2k+1} f_{k+2} \ , \quad (k = 1, 2, \ldots)$$

Then, $0 \leq (T z_k, \hat{K} z_k)_1 = a_{2k+3}(a_{2k+3} a_{2k-1} - a_{2k+1}^2)$
which yields

$$0 \leq (a_{2k-1}/a_{2k+1}) - (a_{2k+1}/a_{2k+3}) \equiv w_k - w_{k+1} \ , \quad (k = 1, 2, \ldots)$$

To prove convergence, we may use (48) to express $w_k = a_{2k+1}/a_{2k-1}$ in the form

$$w_k = \sum_{i=1}^{\infty} |c_i|^2 \lambda_i^{-2k} / \sum_{i=1}^{\infty} |c_i|^2 \lambda_i^{-2(k+1)} \ , \quad (k = 1, 2, \ldots). \tag{51}$$

Using the simplified notation, $\Lambda_i = \lambda_i^2$, and the fact that by hypotesis $c_1 = c_2 = \ldots = c_{r-1} = 0$, we deduce from (51) the expression

$$w_k = \Lambda_r \frac{P(k)}{Q(k)} \ , \quad (k = 1, 2, \ldots), \tag{52}$$

where P(k) and Q(k) are the series

$$P(k) = \sum_{i=r}^{\infty} |c_i|^2 (\Lambda_r / \Lambda_i|)^k, \qquad Q(k) = \sum_{i=r}^{\infty} |c_i|^2 (\Lambda_r / \Lambda_i)^{k+1}$$

From Bessel's inequality $\sum_{i=r}^{\infty} |c_i|^2 \leq ||f_0||_2^2$ and the fact that $(\Lambda_r/\Lambda_i) < 1$ for $i > r$, it follows that the series $P(k)$ and $Q(k)$ are uniformly convergent with respect to the parameter k, and their difference may be expressed in the form

$$P(k) - Q(k) = \sum_{i=r}^{\infty} |c_i|^2 (\Lambda_r/\Lambda_i)^k - (\Lambda_r/\Lambda_i)^{k+1}) \leq \sum_{i=r+1}^{\infty} |c_i|^2 (\Lambda_r/\Lambda_i)^k$$

$$\leq (\Lambda_r/\Lambda_{r+1})^k ||f_0||_2^2 \ , \ (k \geq 1). \tag{53}$$

Since $P(k) \geq Q(k) \geq |c_r|^2 > 0$, it follows from (53) that $[P(k) - Q(k)] \to 0$ and $P(k)/Q(k) \to 1$ as $k \to \infty$. Therefore, from (52) it follows that w_k converges to $\Lambda_r = \lambda_r^2$.

(b) By Proposition 4, Eq.(44), the elements of the sequence $s_k = \lambda_r^{2k} f_{2k}, k = 1, 2, \ldots$ may be represented by the series

$$s_k = \lambda_r^{2k} \sum_{i=r}^{\infty} c_i \lambda_i^{-2k} u_i = \sum_{i=r}^{\infty} c_i (\Lambda_r/\Lambda_i)^k u_i \ , \quad (k = 0, 1, \ldots)$$

convergent in H_1 and H_2-metrics. Hence, due to the orthonormality of the eigenvectors u_i in H_2, $i = 1, 2, \ldots$, it follows that

$$||s_k - c_r u_r||_2^2 = || \sum_{i=r+1}^{\infty} c_i (\Lambda_r/\Lambda_i)^k u_i||_2^2 = \sum_{i=r+1}^{\infty} |c_i|^2 (\Lambda_r/\Lambda_i)^{2k}.$$

Applying Bessel's inequality and the fact that by hypothesis $\Lambda_r < \Lambda_{r+1} \leq \Lambda_{r+2} \leq \ldots$, we obtain the error estimate

$$||s_k - c_r u_r||_2^2 \leq ||f_0||_2^2 (\Lambda_r/\Lambda_{r+1})^{2k} \ , \quad (k = 1, 2, \ldots).$$

Thus, it follows that the sequence $\{s_k\}$ converges in the H_2-metric to an eigenfunction $c_r u_r \in H_2^r$, with the error estimate given above. ■

Now, let us assume that a lower bound l_{r+1} for the eigenvalue $|\lambda_{r+1}|$ can be determined by some method such as, for example, suggested by Lemma 1. Then, using the iterative process (43), we can derive a sequence of lower bounds that converges to $|\lambda_r|$.

Theorem 4. *Assume the hypothesis of Theorem 1. If l_{r+1} is a lower bound for $|\lambda_{r+1}|$ such that for some positive integer N we have $\sqrt{w_N} \leq l_{r+1} \leq |\lambda_{r+1}|$, then*

$$\Lambda_r \geq (l_{r+1}^2 - w_k) w_{k+1} / (l_{r+1}^2 - w_{k+1})$$

for $k \geq N$ and the sequence of lower bounds converges to Λ_r as $k \to \infty$.

Proof. The proof of the above theorem is based on the corresponding results for linear K-real eigenvalue problems (see [5], p.207). ■

Theorems 3-4 allow us to bracket the eigenvalues of a quadratic eigenvalue problem (35) $L_\lambda x = 0$ by a procedure which is similar to the Temple-Lehman method for linear eigenvalue problems $Mu - \lambda Nu = 0$. In that sense the above results may be considered an extension of the Temple-Lehman method to nonlinear (quadratic) eigenvalue problems (35), where A,B,C are symmetrizable operators in H. Important extensions and applications of the Temple-Lehman method to linear problems $Mu - \lambda Nu = 0$, where M and N are partial differential operators, may be found in the work of F. Goerisch and H. Haunhorst [16].

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