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Regular Partially Invariant Submodels of Gas Dynamics Equations

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The Program SUBMODELS [1] is aimed to exhaust all possibilities derived from the symmetry of differential equations for construction of submodels (i.e., systems of equations of the reduced dimension) which describe classes of exact solutions for initial equations. In the frame of this Program, our paper communicates the description of all (up to similarity) regular submodels for the system of Gas Dynamics with the general state equation. Submodels are classified by their types (σ, δ), where σ is a rank and δ is a defect. Regularity of a submodel means that invariant independent variables are functions of initial independent variables only. The classification tables, comments to them, and some characteristic examples are presented.

1 General points

The system E of differential equations with n independent variables $x = (x^1, ..., x^n)$ and m unknown functions $u = (u^1, ..., u^m)$ is considered. Let E admit a local Lie group H of transformations of the space $R^{n+m}(x, u)$ and let H have the *universal invariant* $I = (I^1, ..., I^\ell)$.

Definition 1. The system of equations $E \mid_M$ obtained by the reducing E on the invariant manifold M of the group H is called *H–submodel of the type* (σ, δ) if M has the dimension $n+\delta$ in $R^{n+m}(x, u)$ and the dimension σ in the space of invariants $R^\ell(I)$ thereby $\sigma \geq 0$ and $0 \leq \delta < m$. Solutions for the system $E \mid_M$ are called *partially invariant solutions* of rank σ and defect δ, or *$H(\sigma, \delta)$–solutions* for short.

If such M exists, then components of I may be selected so that with decompositions $u = (u', u'')$, $I = (I', I'')$, where $u' = (u^1, ..., u^{m-\delta}), I' = (I^1, ..., I^{m-\delta})$ the relations hold (g.r. means "general rank")

$$\partial I''/\partial u' = 0, \quad \text{g.r. } \|\partial I'/\partial u'\| = m - \delta, \quad \text{g.r. } \|\partial I''/\partial(x, u'')\| = \sigma, \tag{1.1}$$

$$\sigma = \ell - m + \delta. \tag{1.2}$$

Then if we put

$$v = I'(x, u), \quad y = I''(x, u''), \tag{1.3}$$

the equations of M may be written in the form

$$M:\ v = V(y). \tag{1.4}$$

The equalities (1.3), (1.4) give the *representation* of $H(\sigma,\delta)$–solutions via invariants of the group H. The $H(\sigma,\delta)$–submodel equations are obtained by substitution of this representation in the equations of E. As a result, the system E is divided into two subsystems: *invariant subsystem* E/H for unknown functions $V(y)$ and some additional, in common case overdetermined subsystem Π for "superfluous" functions $u''(x)$. If Π is noncompatible then the set of $H(\sigma,\delta)$–solutions is empty. Therefore, the problem of finding $H(\sigma,\delta)$–solutions in the first turn is persisted in the investigation of compatibility of equations Π (*leading* Π *into involution*).

Definition 2. The number

$$\mu = \text{g.r. } \|\partial I''/\partial u''\|$$

is called the *measure of nonregularity* for $H(\sigma,\delta)$–submodel. For, $\mu = 0$ the $H(\sigma,\delta)$–submodel is called *regular*, otherwise (if $\mu > 0$) *nonregular*.

The significant differencies between regular and nonregular submodels are described in [2]. Particularly for regular solutions, invariant independent variables y (1.3) in the subsystem E/H depend on initial independent variables only that makes the leading subsystem Π into involution essentially easier.

In applications, it is used not the group H itself but its *Lie algebra of operators with the basis*

$$X_\alpha = \xi^i_\alpha(x,u)\partial_{x^i} + \eta^k_\alpha(x,u)\partial_{u^k} \qquad (\alpha = 1,...,r). \tag{1.5}$$

Then the number ℓ is defined by the general rank of the matrix, whose elements are coordinates of the operators (1.5)

$$r_* = \text{g.r. } \|\xi^i_\alpha(x,u), \eta^k_\alpha(x,u)\|,$$

namely, $\ell = n + m - r_*$. The substitution into (1.2) gives

$$\sigma = \delta + n - r_*, \tag{1.6}$$

that determines the rank σ via the given defect δ.

It is known [3] that possible values of the defect δ satisfy inequalities

$$\max\{r_* - n, 0\} \le \delta \le \min\{r_* - 1, m - 1\}. \tag{1.7}$$

It follows from inequalities (1.7) and formula (1.6) that the number of different types (σ,δ) is equal to nm.

Definition 3. $H(\sigma,0)$–solutions are called *invariant* H*–solutions* of the rank σ (always $\sigma < n$).

For invariant H–solutions, it will be $y = I''(x)$ in (1.3), i.e., all invariant H–solutions are regular ones. Here, the submodel $E \mid_M$ consists of the invariant subsystem E/H only, the subsystem Π is empty and there is no problem of leading into involution.

Being defect $\delta > 0$ the process of leading the subsystem Π into involution may be branched and give *different classes* of $H(\sigma,\delta)$–solutions. Some of these classes may occur to be $H_1(\sigma_1,\delta_1)$–solutions for a *subgroup* $H_1 \subset H$. It is known [3] that thereby $\sigma_1 \ge \sigma$, $\delta_1 \le \delta$ always.

Generally speaking, it is easier to find and research solutions with a lower rank and fixed defect or lower defect and fixed rank. Therefore, the following notion is useful.
Definition 4. Let some class of $H(\sigma,\delta)$–solutions be the class of $H_1(\sigma_1,\delta_1)$–solutions too with a subgroup $H_1 \subset H$ so that

$$\sigma_1 = \sigma, \quad \delta_1 < \delta. \tag{1.8}$$

In this case, it is said that *reduction* of the $H(\sigma,\delta)$–solutions *to less defect* for this class takes place. Vice versa, let some class of $H(\sigma,\delta)$–solutions be the class of $H_2(\sigma_2,\delta_2)$–solutions too with an *overgroup* $H_2 \supset H$ so that

$$\sigma_2 < \sigma, \quad \delta_2 = \delta. \tag{1.9}$$

Then it is said that *inverse reduction* of the $H(\sigma,\delta)$–solutions *to less rank* takes place.

There are many examples of reduction of solutions to invariant ones ($\delta = 0$). The theorems of reduction giving sufficient conditions on the basis of structural properties of the subsystem Π are contained in [3].

2 Gas Dynamics equations

The system E is considered on the 9–dimensional *base space* $R^9(t,\vec{x},\vec{u},\rho,S)$ with independent variables t (time), $\vec{x} = (x,y,z)$ (Descartes coordinates in R^3) and unknown functions $\vec{u} = (u,v,w)$ (velocity), ρ (density), S (entropy)

$$\rho D\vec{u} + \nabla p = 0, \quad D\rho + \rho \operatorname{div}\vec{u} = 0, \quad DS = 0, \quad p = F(\rho,S). \tag{2.1}$$

Here $D = \partial_t + \vec{u}\cdot\nabla$, $\nabla = (\partial_x,\partial_y,\partial_z)$. Pressure p is defined by the *state equation* (last ones in (2.1)), where $F(\rho,S)$ is the known smooth function which satisfies inequalities $F_\rho = c^2 > 0$ (c is the speed of sound) and $F_S > 0$.

It is known that system (2.1) admits the 11–parameters local Lie group G_{11} of transformations of the space R^9. The algebra Lie L_{11} of this group has the following basis of operators:

$$\begin{aligned}
&X_1 = \partial_x,\ X_2 = \partial_y,\ X_3 = \partial_z,\\
&X_4 = t\partial_x + \partial_u,\ X_5 = t\partial_y + \partial_v,\ X_6 = t\partial_z + \partial_w\\
&X_7 = y\partial_z - z\partial_y + v\partial_w - w\partial_v,\\
&X_8 = z\partial_x - x\partial_z + w\partial_u - u\partial_w,\\
&X_9 = x\partial_y - y\partial_x + u\partial_v - v\partial_u,\\
&X_{10} = \partial_t,\ X_{11} = t\partial_t + x\partial_x + y\partial_y + z\partial_z.
\end{aligned} \tag{2.2}$$

The normalized optimal system of subalgebras ΘL_{11} is presented in [1]. It consists of 220 representatives being a potential source of $H(\sigma,\delta)$–submodels.

These representatives are denoted by symbols $L_{r,i}$, where r is the dimension of the subalgebra and i is the number of subalgebras of given dimension according to table ΘL_{11}.

3 Types of submodels

For system (2.1), we have $n = 4$ and $m = 5$. So there are possible 20 types of $H(\sigma, \delta)$–submodels for (2.1) a priori. These types are defined by relations (1.6) and (1.7). The result is presented in Table 1. It contains initial information about the number of different submodels of each type. These submodels are defined not only by their types, but depend on a concrete representation of the algebra Lie L_{11} by operators (2.2) in the space R^9. For example, by virtue (2.2), quantities ρ, S, and p are invariants for each subgroup of G_{11}.

TABLE 1.

σ	δ	ℓ	N	N_{reg}	Comments
3	0	8	13 (1)	13 (1)	invariant
2	0	7	26 (2)	26 (2)	invariant
3	1		26 (2)	1 (2)	
1	0	6	38 (3)	38 (3)	invariant
2	1		51 (3)	12 (3)	
3	2		47 (3)	—	
0	0	5	5 (4)	5 (4)	invariant isobaric
1	1		46 (4) + 1 (5)	29 (4)	
2	2		47 (4) + 1 (5)	1 (4)	
3	3		47 (4) + 2 (5)	—	
0	1	4	22 (5) + 2 (6)	22 (5)	partial isobaric
1	2		35 (5) + 13 (6)	9 (5) + 1 (6)	
2	3		35 (5) + 8 (6)	—	
3	4		35 (5) + 8 (6)	—	
0	2	3	13 (6) + 10 (7)	13 (6) + 10 (7)	partial isobaric
1	3		1 (6) + 8 (7)	1 (7)	barochronic
2	4		1 (6) + 5 (7)	—	
0	3	2	1 (≥ 7)	1 (≥ 7)	isobaric
1	4		1 (7)	—	isentropic and barotropic
0	4	1	—	—	no

The types of submodels (σ, δ) are shown in the first and the second columns of the Table 1. The possible dimensions of the space of invariants ℓ are presented in the third one. Information about the number of subalgebras N from the optimal system generating different submodels is in the fourth column. Here, the number N is presented as the sum $N_1(r_1) + N_2(r_2)$, where r_k is the dimension of a subalgebra and $N_k(r_k)$ is the number of r_k -dimensional subalgebras. The number of subalgebras N_{reg} generating different *regular* submodels (in the same designations) is in the fifth column. The sign – means the absence of the subalgebra of this type. The sixth column contains comments about concrete classes of solutions.

Some classes of motions are emphasized among the gas motions described by system (2.1). The list of special submodels is presented below for convenience of future references.

Type (1,4). *Isentropic motions*, $S = \text{const}$. System (2.1) is reduced to the following:

$$\rho D\vec{u} + F'(\rho)\nabla\rho = 0, \quad D\rho + \rho \text{div}\vec{u} = 0, \quad p = F(\rho) \tag{3.1}$$

with the given function $F(\rho)$.

Type (1,4). *Barotropic motions*, $p = P(\rho)$. These motions may be isentropic when $P(\rho) = F(\rho)$, otherwise S is not constant. In the last case, system (2.1) is reduced to

$$D\vec{u} + \nabla e = 0, \quad \text{div}\vec{u} = 0, \quad De = 0, \tag{3.2}$$

where $e = e(\rho)$ is *entalpy*. The pressure p as a function of ρ is defined by the formula $p = \int \rho e(\rho) d\rho$.

Type (1,3). *Barochronic motions*, $p = p(t)$, $\rho = \rho(t)$. These motions are isentropic. System (2.1) has the form:

$$D\vec{u} = 0, \quad \text{div}\vec{u} = -\rho'/\rho, \quad p(t) = F(\rho(t)) \tag{3.3}$$

with the given function $F(\rho)$.

Type (0,3). *Isobaric motions*, $p = \text{const}$. System (2.1) is reduced to

$$D\vec{u} = 0, \quad \text{div}\vec{u} = 0, \quad D\rho = 0, \quad F(\rho, S) = \text{const} \tag{3.4}$$

with the given function $F(\rho, S)$ which defines the dependence $\rho(S)$.

Systems of equations (3.2), (3.3), (3.4) are overdetermined. It is the known expression for a general solution containing arbitrary functions for system (3.4). System (3.3) has been lead into involution, but its general solutions is not constructed now. The problem of leading into involution the system (3.2) is unsolved too.

Quantities ρ, S are invariants of every subalgebra of ΘL_{11}. This fact influences the structure of submodels. In particular, there are no submodels of the type (0,4). Really, rank $\sigma = 0$ means that $\rho = \text{const}$ and $S = \text{const}$, defect $\delta = 4$ means that there are 4 superfluous functions, but only three functions u, v, w remain unknown here. The same fact shows that all submodels of type $(0, \delta)$ describe isobaric motions. All the solutions of the type $(0, 0)$ (constant solution among them) are contained in the class of solutions of type the (1,0) generated by the subalgebra $L_{3,33}$ with the basis X_2, X_3, X_{10}. These solutions have the following general form:

$$u = 0, \quad v = v(x), \quad w = w(x), \quad \rho = \rho(x), \quad p = \text{const}.$$

All invariant submodels (type $(\sigma, 0)$) were described separatly and are not presented in this work (type (3,0) has been published in [1]). So here the following possible regular submodels are presented only, i.e., unique ones of types (3,1), (2,2) and sets of submodels of types (2,1), (1,1), and (1,2). One submodel of type (1,2) is marked. It arises in researching some other submodels and by this reason may by called *canonical* one.

4 Canonical submodel of the type (1,2)

It describes a two–dimensional version of barochronic motions and is generated by any-one of subalgebras $L_{5,17}$, $L_{5,37}$, $L_{6,8^{00}}$. All these subalgebras have identical invariants t, u, ρ, S and superfluous functions v, w. The representation of a solution is

$$(u, \rho, S) \ | \ t \ ; \quad (v, w) \ | \ (t, x, y, z).$$

It follows from (3.3) that $u'(t) = 0$, i.e., $u =$ const. By means of Galilei translation, u can be reduced to zero: $u = 0$. Then equations (3.3) come to

$$v_t + vv_y + wv_z = 0, \quad w_t + vw_y + ww_z = 0, \tag{4.1}$$

$$v_y + w_z = 2h, \tag{4.2}$$

with an unknown function $h = h(t)$. Then the density $\rho = \rho(t)$ is defined from the equation $\rho' = -2h\rho$.

Compatibility conditions for system (4.1), (4.2) are

$$v_y w_z - v_z w_y = k, \tag{4.3}$$

$$k = h' + 2h^2, \quad k' + 2hk = 0. \tag{4.4}$$

Under these conditions, system (4.1) – (4.4) is in involution. This system is integrable and its solution contains arbitrary functions. It is linearized by change of variables

$$z = Z(t, y, v), \quad w = W(t, y, v) \tag{4.5}$$

and is reduced to the following

$$W_v = Z_y + 2hZ_v, \quad W_y = -kZ_v; \tag{4.6}$$

$$W_t + vW_y = 0, \quad Z_t + vZ_y = W. \tag{4.7}$$

The subsystem (4.6) is integrated as a system of equations with constant coefficients. The form of solutions depends on the discriminant $d = h^2 - k$ sign. System (4.6) has a hyperbolic type for $d > 0$, elliptic type for $d < 0$ and parabolic one when $d = 0$. Functions $h(t)$ and $k(t)$ are determined easy as the solutions of ordinary differential equations (4.4). They are rational functions of t. The subsystem (4.7) is reduced to ordinary differential equations too. Finally the solution is defined in implicit form from relations (4.5) with the known functions Z and W.

5 Regular submodel of the type (3,1)

It this generated by the subalgebra $L_{2,26}$ with the basis X_1, X_4. Invariants are t, y, z, v, w, ρ, S and a superfluous function is u. The representation of a solution is:

$$u = u(t, x, y, z); \quad (v, w, \rho, S) \ | \ (t, y, z).$$

Here, system (2.1) takes the form

$$D'u + uu_x = 0, \quad \rho D'v + p_y = 0, \quad \rho D'w + p_z = 0,$$
$$D'\rho + \rho(u_x + v_y + w_z) = 0, \quad D'S = 0, \quad p = F(\rho, S), \tag{5.1}$$

with $D' = \partial_x + v\partial_y + w\partial_z$.

In order to lead system (5.1) into involution, it is sufficient to note that by virtue of the fourth equation in (5.1), u_x is the function of t, y, z only. Therefore, this superfluous function can be presented in the following form:

$$u = (x + X)/h \tag{5.2}$$

with functions $X = X(t, y, z)$, $h = h(t, y, z)$. With representation (5.2), system (5.1) is transformed to

$$\rho D'v + p_y = 0, \quad \rho D'w + p_z = 0, \quad D'\rho + \rho(v_y + w_z) = -\rho/h,$$
$$D'S = 0, \quad D'X = 0, \quad D'h = 1. \tag{5.3}$$

System (5.3) is in involution. It may be treated as a submodel of two–dimensional gas motions with the mass source (right side in the third equation) depending on the solution.

The submodel of two–dimensional gas motions without source can be derived from (5.5) by introducing modified density and pressure $\rho^* = h\rho$, $p^* = hp$. Then a new "equation of state" will depend on h, namely, $p^* = hF(\rho^*/h, S)$.

6 Regular submodel of the type (2,2)

This submodel is generated by the subalgebras $L_{4,47}$ and $L_{5,14}$. They have identical invariants t, x, u, ρ, S and superfluous functions are v, w. Representation of the solution is

$$(u, \rho, S) \mid (t, x)\,; \quad (v, w) \mid (t, x, y, z).$$

System (2.1) after introducing auxiliary invariant function $h = h(t, x)$ is divided into an invariant subsystem

$$\rho(u_t + uu_x) + p_x = 0, \quad \rho_t + u\rho_x + \rho u_x + 2\rho h = 0,$$
$$S_t + uS_x = 0, \quad p = F(\rho, S) \tag{6.1}$$

and overdetermined subsystem for superfluous functions v, w:

$$v_t + uv_x + vv_y + wv_z = 0, \quad w_t + uw_x + vw_y + ww_z = 0,$$
$$v_y + w_z = 2h. \tag{6.2}$$

Compatibility conditions are

$$v_y w_z - v_z w_y = k(t, x),$$
$$h_t + uh_x + 2h^2 = k, \quad k_t + uk_x + 2hk = 0. \tag{6.3}$$

Systems (6.1)–(6.3) are in involution. Subsystem (6.2), (6.3) is similar to the canonical system (4.1)–(4.4) and exactly coincides with it after introducing a Lagrange coordinate $\xi = \xi(t,x)$ as the solution of the equation $\xi_t + u\xi_x = 0$ and changing variables $(t,x) \to (t,\xi)$. Hence, subsystem (6.2), (6.3) is integrable. The remaining subsystem (6.1) describes one–dimensional gas motions with the mass source $2\rho h$.

The function $\xi(t,x)$ can be chosen in such a manner that following relations hold

$$\rho = k\xi_x, \quad \rho u = -k\xi_t. \tag{6.4}$$

They integrate the second equation of (6.1). Here $S = S(\xi)$, so one quazilinear second–order equation with known coefficients can be derived for the function $\xi(t,x)$ (similar to the case for one–dimensional gas motion).

7 Regular submodels of type the (2,1)

There are 12 subalgebras from the optimal system ΘL_{11} generating these submodels accordingly to the Table 1. All these subalgebras $L_{3,i}$ are three–dimensional. We have the detailed description of these submodels. Here we present a brief one in Table 2.

TABLE 2.

i	Basis $L_{3,i}$	Invariants		S. f.	Char. class
		independent	unknown		
6	1, 4, $\alpha 7+11$	$R/t,\ \theta-\alpha\ln t$	$V,\ W$	u	χ^s
8	7, 8, 9	$t,\ r$	$U,\ H$	ω	χ^e
11	1, 4, 7	$t,\ R$	$V,\ W$	u	χ^e
13	2, 3, 7	$t,\ x$	$u,\ q$	φ	χ^e
15^{00}	3 + 5, 2 − 6, 7	$t,\ x$	u, V^*	θ^*	χ^e
17	1, 4, 7 + 10	$R, \theta - t$	$V,\ W$	u	χ^s
23	1, 4, $\alpha 6+11$	$y/t,\ z/t-\alpha\ln t$	$v,\ w-z/t$	u	χ^s
27^{00}	3, 6, 4 + 10	$x-\frac{1}{2}t^2,\ y$	$u-t,\ v$	w	χ^s
29	1, 4, 10	$y,\ z$	v, w	u	χ^s
38^{000}_1	3, 1 + 5, 6	$t,\ x-y/t$	$u,\ v-y/t$	w	χ^e
38^{000}_2	3, 5, 2 + 6	$t,\ x$	$u,\ w+tv-y$	v	χ^e
46	1, 2, 4	$t,\ z$	$v,\ w$	u	χ^e

The numbers i of generating subalgebras $L_{3,i}$ are shown in the first column of Table 2. The basic operators of a subalgebra are presented in the second column. Here the symbolic notation is used: instead of operator X_k, its number k is shown only; the symbol $\alpha 7+ 11$, where α is an arbitrary real number, replaces operator $\alpha X_7 + X_{11}$ and so on. Bases of invariants of subalgebras $L_{3,i}$ are indicated in the third and fourth columns. The following standart designations are used: $r = \sqrt{x^2+y^2+z^2}, R = \sqrt{y^2+z^2}$, $\theta = \arctan(z/y)$, $V = v\cos\theta + w\sin\theta$, $W = -v\sin\theta + +w\cos\theta$, $q = \sqrt{v^2+w^2}$, $\varphi = \arctan(w/v)$. Individual designations are used for $i = 8$, where U and H are radial and tangent (to spheres $r = \text{const}$) components of velocity vector $\vec{u}$, respectively, while ω is the angle between projections of $\vec{u}$ onto sphere and meridian. For $i = 15^{00}$, quantities V^* and θ^* are introduced

$$v = \frac{ty+z}{t^2+1} + V^* \cos\theta^*, \quad w = \frac{tz-y}{t^2+1} + V^* \sin\theta^*.$$

In addition, for all listed submodels, the quantities ρ and S are invariant. They are omitted in this table for short. The superfluous functions (S.f.) are presented in the fifth column. The last column shows additional qualitative peculiarity of submodels, their characteristic class χ^e or χ^s. Submodels of the class χ^e consist of hyperbolic–type equations similar to ones for *one–dimensional unsteady motions.* Submodels of the class χ^s consist of mixed elliptic – hyperbolic–type equations similar to ones for *two–dimensional steady flows.*

The existence of corresponding partially invariant solutions is established for all submodels from Table 2. A submodel for $i = 8$ has been considered in the work [5]. A submodel for $i = 15^{00}$ is reduced to an invariant solution. All other submodels from Table 2 are nonreducible.

8 Regular submodels of the type (1,2)

All these submodels with one exception are generated by five–dimensional subalgebras $L_{5,i}$. The submodel appropriate to the subalgebra $L_{6,10}$ with the basis X_1, X_2, X_3, X_7, X_8, X_9 is exclusive. It has invariants t, $|\vec{u}|$, ρ, S and superfluous functions v, w. This submodel describes special barochronic motions having a constant $|\vec{u}|$:

$$u^2 + v^2 + w^2 = a^2 \qquad (a = \text{const}) \tag{8.1}$$

The existence of these solutions has been established but the problem of finding a general solution for corresponding overdeterminate system (3.3) with additional relation (8.1) is open.

TABLE 3.

i	Basis $L_{5,i}$	Invariants		S.f.	Doubles	
		independent	unknown		$r=5$	$r=6$
7	$1,5,6,\alpha 4{+}7,$ $\beta 4{+}11$	R/t	$u{-}\alpha\varphi{-}$ $-\beta \ln t$	q,φ		
10	$2,3,5,6,$ $\beta 4{+}7{+}\alpha 11$	$x/t-$ $-(\beta/\alpha)\ln t$	$u - x/t$	v,w	26	6^0
13	$2,3,5,6,$ $\beta 4+7$	t	$u - x/t$	v,w	35	15
15	1, 2, 3, 4, 7	t	q	u,φ		
16	$1,4,3+5,$ $2-6,7$	t	V^*	u,θ^*		
17	2, 3, 5, 6, $1+7$	t	u	v,w	37	8^{00}
18	$2,3,5,6,$ $\beta 4{+}7{+}\beta 10$	$x - \frac{1}{2}t^2$	$u - t$	v,w	31	12^{00}
19	2, 3, 5, 6, $7+10$	x	u	v,w	33	13
36	2, 3, 4, 5, $1+6$	t	$w{-}tu{-}x$	u,v		

TABLE 4.

i	Basis $L_{4,i}$	Invariants independent	Invariants unknown	S.f.
1	7, 8, 9, 11	r/t	U, H	ω
4	$1, 4, 10, 7+\alpha 11$	$Re^{-\alpha\theta}$	$q, \varphi-\theta$	u
5^0	$5, 6, 7, \beta 4+11$	$x/t-\beta\ln t$	$u-x/t, q^*$	φ^*
6	1, 4, 7, 11	R/t	$q, \varphi-\theta$	u
7^0	$2, 3, 7, \beta 4+11$	$x/t-\beta\ln t$	$u-x/t, q$	φ
9^0	$1, 5, 6, \beta 4+7$	t	$u-\beta\varphi^*, q^*$	φ^*
10^0	2, 3, 4, 7	t	$u-x/t, q$	φ
12	$1, 2, 3, \beta 4+7$	t	$u-\beta\varphi, q$	φ
13	7, 8, 9, 10	r	U, H	ω
14	2, 3, 7, 10	x	u, q	φ
16^0	2, 3, 7, 4 + 10	$x-\frac{1}{2}t^2$	$u-t, q$	φ
17	4, 5, 6, 7	t	$u-x/t, q^*$	φ^*
18	4, 5, 6, 1 + 7	t	$u+(\varphi^*-x)/t, q^*$	φ^*
19	$4, 3+5, 2-6, \alpha 1+7$	t	$u+(\alpha\theta^*-x)/t, V^*$	θ^*
20	$1, 3+5, 2-6, \alpha 4+7$	t	$u-\alpha\theta^*, V^*$	θ^*
21	2, 3, 4, 1 + 7	t	$u+(\varphi-x)/t, q$	φ
23	1, 4, 10, 11	z/y	v, w	u
29	$1, 4, 6, \alpha 5+11$	$y/t-\alpha\ln t$	$v-y/t, w-z/t$	u
30^0	$2, 3, 6, \beta 4+\sigma 5+11$	$x/t-\beta\ln t$	$u-x/t, v-\sigma\ln t$	w
35^0	$2, 3, 5, 4+\beta 6+10$	$x-\frac{1}{2}t^2$	$u-t, w-\beta t$	v
36^0	2, 3, 5, 6 + 10	x	$u, w-t$	v
38	2, 3, 5, 10	x	u, w	v
41	$1, \sigma 2+\tau 3+4,$ $\alpha 3+5, \beta 2+6$	t	j_1, j_2	u
42	1, 4, 3 + 5, 2 - 6	t	V^*, θ^*	u
43	1, 4, 5, 6	t	$v-y/t, w-z/t$	u
44	$2, \alpha 1+3, 1+5, 6$	t	$u, v-\alpha tw-x+\alpha z$	w
46	$2, \alpha 1+3, 5, 6$	t	$u, w+(x-\alpha z)/\alpha t$	v
48	1, 2, 3 + 5, 6	t	$u, v+tw-z$	w
50	1, 2, 3, 4	t	v, w	u

The full list of generating subalgebras $L_{5,i}$ is presented in Table 3 similar to Table 2. The important *effect of doubling* takes place here: two (or more) nonsimilar subalgebras generate the identical submodels because they have identical universal invariants. This is a consequence of the special representation of the Lie algebra L_{11} by operators (2.2). Effect of doubling is taken into account in Table 3. All the numbers i of doubling $L_{5,i}$ and $L_{6,i}$ subalgebras are shown in the last two additional columns of Table 3.

The submodel for $i = 17$ is a canonical one (see n^0 4). Submodels for $i = 10, 13, 18, 19$ contain subsystems similar to the canonical.

9 Regular submodels of the type (1,1)

All 29 submodels of this type are generated by four–dimensional subalgebras $L_{4,i}$. Their brief description is presented in Table 4, which is built on the same principles and with the same designations like Tables 2, 3. Additional invariants q^*, φ^*, j_1, j_2 occuring here are defined by the following relations

$$v = y/t + q^* \cos\varphi^*, \quad w = z/t + q^* \sin\varphi^*;$$

$$j_1 = (t^2 - \alpha\beta)v + (\sigma t - \beta\tau)u - ty + \beta z,$$
$$j_2 = (t^2 - \alpha\beta)w + (\tau t - \alpha\sigma)u - \alpha y + tz.$$

Corresponding $H(1,1)$–solutions exist for all submodels from Table 4. Submodels with t as the independent variable describe the special barochronic gas motions.

10 Concluding remarks

A complete list of all 100 regular partially invariant solutions of Gas Dynamics equations (2.1) with the general state equation $p = F(\rho, S)$ is the result of the given investigation. This list may be wider for special state equations according to classification of "big" models of Gas Dynamics [1].

There are many enough nonregular submodels for system (2.1) too. They will be subjects for a future investigation. As a rule, these submodels exist for the special state equations only. The existence problem is not trivial in these cases. It is connected with a problem of leading overdeterminate systems into involution. In [6, 7], this problem is discussed among others for partially invariant submodels of the types (2,2) and (3,3) which are generated by the subalgebra $L_{4,40}$ with the basis X_1, X_2, X_3, X_{10}.

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