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On Asymptotic Methods in the Theory of Differential Equations of Mathematical Physics

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In 1807 Fourier suggested a original method of solving partial differential equations. The method is known to lead to ordinary differential equations containing some arbitrary parameter. Making use of the method, two main problems appear:

1. Finding the solutions (eigenfunctions) of the differential equations obtained.
2. Expansion of a function into the series in terms of eigenfunctions. These two problems in the particular cases were already solved by Fourier.

A great progress in this trend was made by Liouville's works (1838), where an expansion of an arbitrary function into the series in terms of eigenfunctions of the equation

$$\frac{d^2y}{dx^2} + (\lambda g(x) - r(x))y = 0, \tag{1}$$

(λ is a parameter) with certain boundary conditions was considered.

Eigenfunctions obtained by Liouville for Eq.(1) have the property of orthogonality. So, he solved the problems 1 and 2 directly using some Sturm's results.

After appearing works of Fourier, Sturm and Liouville, the theory of representation of the solutions in the form of asymptotic formulae in the parameter λ, when $\lambda \to \infty$, or, that is the same, the asymptotic integration theory, became to develop very quickly.

An important place in the theory is taken by the Poincaré work "New methods in celestial mechanics" (1888), where the practical application of formal series utilized in the asymptotic representation of solutions to differential equations was first demonstrated. Let us illustrate that by means of an example of the second-order differential equation introduced in [1].

Let us consider the differential equation

$$\frac{d^2y}{dt^2} + \left(1 - \frac{a}{t^2}\right) y = 0, \tag{2}$$

where a is a complex number, $t > 0$ is a real variable.

Eq.(2) contains a varying coefficient and so it can not be integrated. Its solution will be sought in the form of a series

$$y(t) = e^{it}\left(C_0 + C_{-1}t^{-1} + \cdots + C_{-n}t^{-n} + \cdots\right). \tag{3}$$

Then by a certain choice of the series coefficients in (3), the function $y(t)$ can formally satisfy Eq.(2) in the sense that when substituting it in Eq.(2), we obtain the identity, in which coefficients of t^{-m} $(m = 0, 1, 2, \ldots)$ are equal to zero. But series (3) is a divergent one. And simultaneously, as Poincaré proved, for large values of t $(t \to \infty)$ an exact solution to Eq.(2) is represented in the form of the asymptotic formula

$$y(t) = e^{it}\left(S_n(t) + O\left(t^{-n-1}\right)\right), \tag{4}$$

where $S_n(t)$ is the n-th partial sum of series (3).

Formal solutions in the form (3) which have the property (4) is said to be asymptotic solutions of the differential equations under consideration and the methods by means of which one can obtain solutions is said to be asymptotic ones.

Note that in Eq.(2) the parameter is missing and asymptotic formula (4) is constructed in the independent variable t $(t \to \infty)$. As to the previous investigations, their methods make possible to built asymptotic solutions in the parameter λ $(\lambda \to \infty)$ for all $x \in [a, b]$, although it is naturally to ask a question on building asymptotic solutions with two variables: the independent variable and the parameter. Such investigations have been carried out by mathematicians. For instance, original results were obtained by M.V. Fedoryuk and elucidated in his doctor theses [2]. But we do not concern these investigations here. The main attention is given to the methods of constructing asymptotic solutions for both large and small parameters.

A great contribution to developing asymptotic methods has been made by the Russian mathematician V.A. Steklov. Proving the "completeness" theorem for eigenfunctions of the Sturm–Liouville equation (Eq.(1)), the scientist solved two problems 1) and 2) on expansion into a series in terms of eigenfunctions of the equation with the same degree of accuracy as for ordinary trigonometric Fourier series.

But all investigations mentioned concern mostly a generalization of Liouville's results, and, therefore, yield selfadjoint differential equations and systems of them. Such quite essential restrictions were removed in the works of L. Schlesinger, G. Birkhoff, Ya. Tamarkin. Thus, Birkhoff derived asymptotic formulae for solving a differential equation of the type

$$y^{(n)} + \rho a_{n-1}(x, \rho) y^{(n-1)} + \cdots + \rho^n a_0(x, \rho) y = 0, \tag{5}$$

where $a_i(x, \rho)$ $(i = 0, 1, \ldots, n-1)$ are analytical functions of the complex parameter ρ $(|\rho| \to \infty)$, which are infinitely differentiable with respect to the real variable $x \in [a, b]$.

As distinct from Schlesinger [3] who has proved the asymptotic properties of solutions only for some fixed ray $\arg \rho = \alpha$ with $|\rho| \to \infty$, Bikhoff [4] has proved the asymptotic properties of solutions for some sector of the complex plane $\theta \leq \arg \rho \leq \eta$.

Schlesinger's and Birkhoff's results have been generalized by Tamarkin [5] for the systems of differential equations of the form

$$\frac{dy_i}{dx} = \sum_{j=1}^{n} a_{ij}(x, \rho) y_j, \tag{6}$$

where $a_{ij}(x, \rho)$, $i, j = \overline{1, n}$ are analytical functions of the complex parameter ρ in a neighborhood of the point $\rho = \infty$, which have a singularity (the pole of order $r \geq 1$) at this point and are infinitely differentiable with respect to the real variable x on the segment $[a, b]$.

Ya.D. Tamarkin [5] has obtained asymptotic formulae for the solutions of system (6) with the parameter ρ in the case, where the roots of the characteristic equation

$$det\ (a_{ij}^{(0)}(x) - \lambda\delta_{ij}) = 0,\ i,j = \overline{1,n} \tag{7}$$

are simple for all $x \in [a,b]$. Here $a_{ij}^{(0)}(x)$ are free terms in the expansions in the parameter ρ; δ_{ij} are Kronecker symbols.

Only for the second-order equation

$$y'' + \rho a_1(x,\rho)y' + \rho^2 a_2(x,\rho)y = 0, \tag{8}$$

he has succeeded in obtaining two formal solutions, when characteristic equation (7) has two equal roots on the segment $[a,b]$. For differential equations of higher order and also for differential systems of type the (6), the case with multiple roots of the characteristic equation has remained not investigated for a long time, and only in the 60–70s in M.I. Shkil's works (who is the author of this review) and, in particular, in his doctor theses [6], the cases with multiple roots of the characteristic equation have been studied in detail. These results and also those obtained by followers of M.I. Shkil are discussed below.

Note that to represent formal solutions of the systems of differential equations containing a large parameter λ by analogy with the well-known Euler method of solving a linear system of differential equations with constant coefficients, one makes use of the expression

$$\exp\left(\rho^r \int\limits_a^x \sum_{s=o}^{r-1} \rho^{-s}\lambda_s(\tau)d\tau\right) \sum_{s=0}^{\infty} \rho^{-s} y_s(x), \tag{9}$$

from which one obtains asymptotic formulae for the solution of the form

$$y(x,\rho) = \exp\left(\rho^r \int\limits_a^x \sum_{s=o}^{r-1} \rho^{-s}\lambda_s(\tau)d\tau\right) \left(\sum_{s=0}^{m} \rho^{-s} y_s(x) + O(\rho^{-m})\right). \tag{10}$$

Asymptotic representation of the solutions in the more general form

$$y_i(x,\rho) = \sum_{k=1}^{n} z_k(x,\rho) \left(\sum_{j=1}^{m} y_{ij}(x)\rho^{-j} + O(\rho^{-m})\right), \tag{11}$$

where $z_k(x,\rho)$ $(k = \overline{1,n})$ are functions satisfying some system of linear differential equations, is adduced in the works of V.S. Pugachov [7].

Pugachov's results have been used by I.M. Rapoport to build the linear transformations, by means of which one can reduce some class of differential equations to a differential system of the L-diagonal type. For solutions of these systems, the asymptotic formulae with respect to the independent variable t in the neighborhood of the point $t = +\infty$ were obtained (1954). Pugachov's ideas have been making use of also by S.F. Feshchenko [8] in asymptotic splitting the system of linear differential equations with slowly varying coefficients.

The works of F. Hukuhara (1934–1936) and G. Territin (1957) are devoted to asymptotic representations of solutions, where the asymptotic splitting of the systems of linear

differential equations with a large parameter into subsystems of smaller order is adduced [9]. Essential results of asymptotic integration of differential equations have been obtained in works of V. Vazov (1968), L. Chezary (1964), R. Langer (1931), where, in particular, an extraordinary complicated case related to furning points is investigated [10–12].

Our short review on asymptotic representation of solutions of differential equations containing a parameter would not be completed enough if we omit the Krylov–Bogolyubov–Mitropolskii methods in nonlinear mechanics [13]. The methods were developed at first by N. Krylov and M. Bogolyubov (1937) for approximate integrating the nonlinear differential equations of the form

$$\frac{d^2x}{dt^2} + \omega^2 x = \varepsilon f\left(x, \frac{dx}{dt}, \varepsilon, t\right), \tag{11$'$}$$

where ε is a small parameter.

Note that equation (11$'$) is of a quite different form than the equations considered previously. Really when $\varepsilon = 0$, (11$'$) becomes a second-order linear differential equation with the constant coefficient ω^2, which is elementary integrable. So, the Poincaré method [14] can be used for equation (11), by which the solution is found in the form of series in powers of the small parameter ε. But in this way, approximate solutions are obtained, which have so–called secular terms (terms in which the independent variable is in front of sine and cosine). This fact constricts considerably the temporal interval t during which one can observe an ossillatory process.

Approximate formulae obtained by the Krylov–Bogolyubov method do not contain secular terms and it makes possible to investigate the ossillatory process on the finite but large enough interval of varying the time t (of order $O(\frac{1}{\varepsilon})$).

Making use of the Krylov–Bogolyubov method, Yu.A. Mitropolskii studied a nonstationary ossillatory processes in systems with one and a few degrees of freedom. He first investigated the rather complicated phenomenon of passing the system through internal and external resonances and published his fundamental results in the monograph [15]. Mitropolskii's results have been generalized by his disciples, in particular, A.M. Samoilenko, to other classes of differential equations.

Under influence of the Krylov–Bogolyubov–Mytropolskii ideas, the theory of asymptotic representation of solutions to linear differential equations with slowly varying coefficients became to intensively develop. And the subject of our investigations is the differential equations with slowly varying coefficients and asymptotic methods of their solution.

Function $f(t)$ is said to be a slowly varying one on the segment (a, b) if its derivative $f'(t)$ is commensurable with some small parameter $\varepsilon \in (0, \varepsilon_0)$ (ε_0 is a real small enough number, $\varepsilon_0 \ll 1$).

It is clear that any function differentiated on the segment (a, b) is a slowly varying one of t if it depends on variable $\tau = \varepsilon t$. Really, in this case we have

$$\frac{df(\tau)}{dt} = \varepsilon \frac{df(\tau)}{d\tau}. \tag{12}$$

The variable τ is said to be a "slow time" as well.

Differential equations in which the coefficients are slowly varying functions are said to be those with slowly varying coefficients. Many differential equations of different types, in particular, the equations considered by Liouville, Schlesinger, Birkhoff and Tamarkin,

can be reduced to such differential equations. We show this by the example of the Sturm–Liouville equation (1). For this purpose we put

$$x = \tau = \varepsilon t, \qquad \varepsilon = \frac{1}{\sqrt{\lambda}}. \tag{13}$$

Then we obtain the equation

$$\frac{d^2y}{dt^2} + (g(\tau) - \varepsilon^2 r(\tau))y = 0 \tag{14}$$

with slowly varying coefficients.

Many mathematical works are devoted to investigating differential equations, in which the coefficient of the higher derivative term is a small parameter. These equations can be also reduced to differential equations with slowly varying coefficients. Let us consider, for example, the differential equation

$$\varepsilon\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + g(x)y = 0, \tag{15}$$

where $\varepsilon > 0$ is a small parameter; $p(x)$, $g(x)$ are functions of the variable $x \in [a, b]$.

The equation is not integrable exactly. But if we put $\varepsilon = 0$, then we have a so-called singular equation that is integrable in quadratures.

Eq.(15) is said to be nonsingular. It is natural to ask a question: how are solutions of the nonsingular equation with small values of the parameter ε connected with solutions of a singular one, in particular, does the solution y_ε of Eq.(15) with $\varepsilon \to 0$ approach the solution y_0 of the singular one? Such problems were investigated by a great number of mathematicians. Fundamental results were first obtained for linear differential equations by I.S. Gradshtein (1950) and for nonlinear ones by A.M. Tikhonov (1948). They determined that $y_\varepsilon \to y_0$ not always with $\varepsilon \to 0$. For this statement to be true, it is necessary to impose some constraints on coefficients of the nonsingular equation and on the initial conditions. If the constraints are not fulfilled then it is necessary to integrate the nonsingular equation at least approximately.

Let us show that Eq.(15) can be reduced to an equation with slowly varying coefficients. For this matter it is sufficient to put

$$x = \tau = \varepsilon t$$

and we arrive at the equation

$$\frac{d^2y}{dt^2} + p(\tau)\frac{dy}{dt} + \varepsilon g(\tau)y = 0 \tag{16}$$

with slowly varying coefficients.

We give our attention only to two types of differential equations which are reduced to differential equations with slowly varying coefficients by means of changing the independent variable. It is possible to point out a number of types (see, e.g., [16]) of differential equations which can be reduced to those with slowly varying coefficients.

The systematic investigation of linear differential equations with slowly varying coefficients has begun from 50s by works of Feshchenko S.F. [17]. He proposed an asymptotic method of representating solutions to the differential equation of the type

$$\frac{d^2y}{dt^2} + \varepsilon p(\tau,\varepsilon)\frac{dy}{dt} + q(\tau,\varepsilon)y = \varepsilon f(\tau,\varepsilon)e^{i\theta(t,\varepsilon)}, \tag{17}$$

where $p(\tau,\varepsilon)$, $q(\tau,\varepsilon)$, $f(\tau,\varepsilon)$, $k(\tau) = \dfrac{d\theta(t,\varepsilon)}{dt}$ are slowly varying functions of the variable $\tau = \varepsilon t$; $\varepsilon > 0$ is a small parameter.

There was considered the case where the function $k(\tau)$ (an external part) for some $\tau \in [0, L]$ becomes such that is equal to one of simple roots (fundamental frequencies) of the characteristic equation corresponding to Eq.(17). This case often occurs in problems of mathematical physics. In mechanics, it is called a resonance. Feshchenko proved an asymptotic character of the solutions obtained in the same sense as in the Krylov–Bogolyubov–Mitropolskii methods, namely, the m–partial sums of these solutions approach the exact solution not when m increases, but when $m \geq 1$ is fixed and $\varepsilon \to 0$.

Feshchenko carried out analogous investigations for a system of differential equations of the type

$$A(\tau,\varepsilon)\frac{d^2x}{dt^2} + \varepsilon C(\tau,\varepsilon)\frac{dx}{dt} + B(\tau,\varepsilon)x = f(\tau,\varepsilon)e^{i\theta(t,\varepsilon)}, \tag{18}$$

where matrices $A(\tau,\varepsilon)$, $C(\tau,\varepsilon)$, $B(\tau,\varepsilon)$ and vector $f(\tau,\varepsilon)$ are represented by power series in the small parameter ε. Here it was supposed that matrices $A_0(\tau)$, $B_0(\tau)$, $C_0(\tau)$ (free terms in general expansions) are symmetric. This strong enough requirement has been removed later in the works of Shkil and his disciples.

Feshchenko has proved a number of extraordinary important theorems on splitting a system of differential equations of the type

$$\frac{dx}{dt} = A(\tau,\varepsilon)x \tag{19}$$

into subsystems of a lower order.

Feshchenko' theorems include, as a particular case, the results of Schlesinger, Birkhoff and Tamarkin on asymptotic representation of solutions to linear differential systems in the case, where roots of the characteristic equation are simple.

But with the help of the asymptotic splitting theorems one can reduce the order of a system only approximately. In the general case, for example, where some roots of the characteristic equation are multiple, while using these theorems, we can't obtain solutions of the system of differential equations in question. But this case occurs often enough both in investigating theoretical questions and solving practical problems. So, already in studying a simple question, that is the Sturm–Liouville equation, one has to deal with multiple roots. They occur in the investigation of differential equations with a small parameter in the part of derivative terms, in optimal control problems and so on. Let us note that the case of multiple roots, especially where multiple roots correspond to multiple elementary divisors, is extraordinary complicate. It is explained by that, generally speaking, the initial system of differential equations has no solutions admitting an expansion in integer powers of the parameter ε. Such solutions, as opposed to the case of somple roots of the characteristic equation, are represented by formal series in different fractional

powers of the parameter with exponents depending not only on multiplicity of the roots of the characteristic equation but also on miltiplicity of elementary divisors corresponding to them.

The case of multiple roots of the characteristic equation has been studied comprehensively by M.I. Shkil. In 60s he obtained essentially new mathematical results on asymptotic representation of solutions to systems of differential equations with slowly varying coefficients. We focus our attention to his some results.

Let the characteristic equation

$$\det\left(A_0(\tau) - \lambda E\right) = 0 \tag{20}$$

(E is an identity matrix) have at least one root $\lambda = \lambda_0$ of constant multiplicity k ($2 \leq k < n$), corresponding to one elementary divisor with the same multiplicity. Then the following theorem is true.

Theorem 1 *If $A(\tau, \varepsilon)$ has on the segment $[0, L]$ all derivatives with respect to τ and the matrix*

$$C(\tau) = T^{-1}(\tau)\left(\frac{dT(\tau)}{d\tau} - A_1(\tau)T(\tau)\right), \tag{21}$$

where $T(\tau)$ is the matrix that reduces $A_0(\tau)$ to the Jordan form, is such that its element

$$C_{k,1}(\tau) \neq 0, \ \forall \tau \in [0, L], \tag{22}$$

then the system of differential equations (19) has a formal solution of the form

$$x(t, \varepsilon) = U(\tau, \mu)\exp\left(\int_0^t \lambda(\tau, \mu)dt\right), \tag{23}$$

where the n-dimensional vector $U(\tau, \mu)$ and the scalar function $\lambda(\tau, \mu)$ have the expansion

$$U(\tau, \mu) = \sum_{s=0}^{\infty} \mu^s U_s(\tau), \quad \lambda(\tau, \mu) = \lambda_0(\tau) + \sum_{s=1}^{\infty} \mu^s \lambda_s(\tau), \quad \mu = \varepsilon^{1/k}. \tag{24}$$

The case, where $C_{k,1} = 0$, has been also studied but

$$C_{k-1,1}(\tau) + C_{k,2} \neq 0, \ \forall \tau \in [0; L]. \tag{25}$$

Then system (19) has a formal solution of the type (23) where $U(\tau, \mu)$, $\lambda(\tau, \mu)$ are represented by formal series with $\mu = \varepsilon^{1/(1-k)}$, and one formal solution is expanded in integer powers of the parameter ε.

Let us adduce the more general result obtained by Shkil for system (19) in the case of multiple roots of the characteristic equation.

Let the following conditions be valid:

1) matrix $A(\tau, \varepsilon)$ has all derivatives with respect to τ on the segment $[0, L]$;

2) characteristic equation (20) has one root $\lambda = \lambda_0(\tau)$ of a constant multiplicity n;

3) the root $\lambda_0(\tau)$ corresponds to $r \geq 1$ elementary divisors of the type

$$(\lambda - \lambda_0(\tau))^{k_1}, \ldots, (\lambda - \lambda_0(\tau))^{k_r}; \tag{26}$$

4) one of the formulae

$$\begin{array}{l} a)\ k_1 = k_2 = \ldots = k_r = k, \\ b)\ k_1 > k_2 > \ldots > k_r. \end{array} \tag{27}$$

is satisfied. Then the following theorem is valid.

Theorem 2 *Let the conditions 1–4a be satisfied. Then for the vector*

$$x = U(\tau, \mu) \exp\left(\int_0^t \lambda(\tau, \mu) dt\right), \tag{28}$$

where the n–dimensional vector $U(\tau, \mu)$ and the scalar function $\lambda(\tau, \mu)$ are represented by the formal series

$$U(\tau, \mu) = \sum_{s=0}^{\infty} \mu^s U_s(\tau), \qquad \lambda(\tau, \mu) = \sum_{s=0}^{\infty} \mu^s \lambda_s(\tau), \tag{29}$$

with $\mu = \varepsilon^{\frac{1}{k}}$, to be a formal vector–solution of system (19) it is necessary and sufficient that the function $(\lambda_1(\tau))^k$ for all $\tau \in [0, L]$ be a root of the equation

$$\det \left\| \begin{array}{l} \rho + C_{k,1}(\tau) C_{k,k+1}(\tau) \ldots C_{k,l_{r-1}+1}(\tau) \\ C_{2k,1}(\tau)\rho + C_{2k,k+1}(\tau) \ldots C_{2k,l_{r-1}+1}(\tau) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ C_{n,1}(\tau) C_{n,k+1}(\tau) \ldots \rho + C_{n,l_{r-1}+1}(\tau) \end{array} \right\| = 0, \tag{30}$$

where $C_{k,1}(\tau)$,...,$C_{n,l_{r-1}+1}(\tau)$; $l_{r-1} = (r-1)k$ are elements of matrix (21).

The proof of the sufficient condition of the theorem gives also an algorithm of obtaining the solution to be found.

An analogous theorem is valid for the case 4b. For the both cases, formal solutions have been proved to be asymptotical expansions in the parameter ε of the exact solutions of system (19). These results have been generalized on the case, where multiple roots of the characteristic equation correspond to the both multiple and simple elementary divisors.

The mathematical results obtained by Feshchenko and Shkil were published as the common monograph, with L.D. Nikolenko being its co-author [17].

The asymptotic expansion theory developed for finite–dimensional spaces has been generalized for infinite-dimensional spaces by Yu.L. Daletsky and S.G. Krein [18].

It is worth to recall the classical works of Kh.L. Territin [19], who built the linear transformations, with the help of which the case of multiple roots of the characteristic equation can be reduced to that of simple roots, and the investigations of V.Vasov, E. Koddington and N. Levinson that simplified maximally the problems with multiple roots as well.

Essentially new methods of asymptotic integration of differential equations with a small parameter in the higher derivative term have been also developed by M.I. Vyshyk and L.A. Lyusternik [20], A.N. Tikhonov, A.B. Vasyl'eva. In particular, Vyshyk and Lyusternik established that the solutions of linear, singularly disturbed problems contain functions of the so-called boundary-layer type and worked out the algorithm of building asymptotic solutions. Developing their ideas, Vasyl'eva worked out the method of building

asymptotic solutions of singularly disturbed problems which is especially effective concerning nonlinear equations [22].

A separate method of the asymptotic integrating of singularly disturbed equations on the finite interval of varying the independent variable was developed by S.A. Lomov in 70s, who proposed the so-called regularization method [23]. Its essence is to replace the singular problem by a regular one via transition to the space of larger dimension. Making use of the method, Lomov and his disciples obtained a set of new results on the asymptotic integration of linear systems.

Concentrating on no details, note that in the last years are intensively developed asymptotic methods of integraton of linear differential systems of the type

$$\varepsilon B(t,\varepsilon)\frac{dx}{dt} = A(t,\varepsilon)x. \tag{31}$$

Some results of these investigations and also ones related to the presence of turning points in the system are adduced in the monograph of M.I. Shkil, I.I. Starun, V.P. Yakovets [24].

It should be noted that the mentioned above results concerning the asymptotic integration of systems of differential equations have been obtained under the certain rather essential condition: roots of the characteristic equation have either to be simple or keep a constant multiplicity on the given interval of the independent variable. If these conditions are violated (turning points appear) then the results obtained before become incorrect. A construction of even formal solutions of the differential systems in question is considerably complicated. More or less complete results under appearance of turning points have been obtained only for one second-order differential equation and systems of two first-order differential equations.

M.I. Shkil has found sufficient conditions of existing formal solutions to systems of differential equations when turning points are presented. But in contract to the previous results, where formal series are pure expansions in some powers of the parameter ε, when turning points are present, coefficients of the formal series depend in turn on the parameter.

Let us formulate M.I. Shkil's result about the system of first-order differential equations of the type

$$\varepsilon^{1/r}\frac{dx}{dt} = A(t,\varepsilon)x, \tag{32}$$

where $A(t,\varepsilon)$ is a matrix of $(n\times n)$-order, $r\geq 1$ is a positive integer.

Theorem 3 *If the following conditions are true:*

1) $A(t,\varepsilon)$ has the expansion

$$A(t,\varepsilon) = \sum_{s=0}^{\infty}\varepsilon^s A_s(t);$$

2) matrices $A_s(t)$ are infinitely differentiable on the segment $(-\infty,+\infty)$;

3) there exists an integer $k\geq 0$ such that roots of the equation

$$\det\left(A_0(t)+\varepsilon A_1(t)+\cdots+\varepsilon^k A_k(t)-\lambda E\right)=0 \tag{33}$$

are simple for $\forall t \in (-\infty, +\infty)$ *and* $\varepsilon \in (0, \varepsilon_0]$*;*
then there exists a formal vector-solution of system (32) of the form

$$x(t, \varepsilon, h) = U(t, \varepsilon, h)\psi(t, \varepsilon, h), \tag{34}$$

where $U(t, \varepsilon, h)$ *is an* $(n \times n)$*-matrix and* $\psi(t, \varepsilon, h)$ *is an* n*-dimensional vector determed by the system of differential equations*

$$h\frac{d\psi(t, \varepsilon, h)}{dt} = \Lambda(t, \varepsilon, h)\psi(t, \varepsilon, h), \tag{35}$$

in which $\Lambda(t, \varepsilon, h)$ *is a diagonal* $(n \times n)$*-matrix represented similarly to the matrix* $U(t, \varepsilon, h)$ *by the formal expansions:*

$$\begin{aligned} U(t, \varepsilon, h) &= \sum_{s=0}^{\infty} h^s U_s(t, \varepsilon), \\ \Lambda(t, \varepsilon, h) &= \sum_{s=0}^{\infty} h^s \Lambda_s(t, \varepsilon), \quad h = \varepsilon^{1/r}. \end{aligned} \tag{36}$$

References

[1] Koddigton E. A., Levinson N., Theory of Ordinary differential equations, 1958, Moscow, Foreign lit. printing, 474p. (in Russian)

[2] Fedoryuk M. V., Asymptotic methods in the theory of linear ordinary differential equations, Moscow, 1966, 24p.

[3] Schlesinger L., Vorlesungen über Differntialgleichungen, *Tenbner. Leipzig,* Berlin, 1908, V.12, S.46.

[4] Birkhoff G.D., On the asymptotic character of the solutions of certain linear differential equations containing a parameter, *Trans. Amer. Math. Soc.,* 1908, V.9, 219–231.

[5] Tamarkin Yu. D., On some general questions of the theory of differential equations and on expansion of a function in series, 1917, 308p.

[6] Shkil M. I., On some asymptotic methods in theory of differential equations with slowly varying coefficients: doctor dissertation, Kyiv, 1968, 420p. (in Russian)

[7] Pugachov V. S., On asymptotic representations of the integral of a linear system of differential equations containing a parameter, *Mat. sborn. N.S.,* 1944, V.15, N 1, 13–46.

[8] Feshchenko S. F., On asymptotic splitting system of linear differential equations, *Ukr. Math. J.,* 1955, V.7, N 2, 167–179; N 4, 252–263.

[9] Hukuhara, Sur les pants singuliers des equation differentielles linealres, *T. Fac. Scl.*, Univ. Hokkaido Math, 1934–1936, V.2, 13–81.

[10] Vasov V., Asymptotic expansions of solutions to ordinary differential equations, Mir, Moscow, 1968, 464p.

[11] Chesary A., Asymptotic behavior and stability of solutions, Mir, Moscow, 1964, 477p.

[12] Langer R. E., Turning points in the linear asymptotic theory, *Bol. Soc. Math. Mech.,* 1960, V.2, N 5, 1–12.

[13] Bogolyubov N. N., On some statistical methods in mathematical physics, Kyiv, A. S. Ukraine printing, 1945, 137p.

[14] Poincaré H., Sur les equations de la physique et mathematique, *Rend. Pal.,* 1894, 10–11, 57–156.

[15] Mitropolskii Yu. A., Problems of asymptotic theory of nonstable fluctuations, Nauka, Moscow, 1964, 432p.

[16] Shkil M. I., Asymptotic methods in differential equations, Vyshcha shkola, Kyiv, 1971, 228p.

[17] Feshchenko S. F., Shkil M. I., Nikolenko L. D., Asymptotic methods in the theory of linear differential equations, American Elsvier Publ. Comp., New York, 1967, 270p.

[18] Daletski Yu.L. and Krein S.G., On differential equations in Hilbert space, *Ukr. Math. J.,* 1950, V.2, N 4, 71–91.

[19] Territin Kh. L., Asymptotic expansion of solutions to system of ordinary differential equations containing parameter, Matematika: Sb. per., 1957, V.1, N 2, 29–59.

[20] Vyshchyk M. I., Lyusternik L. A., Regular singularization and a boundary layer for linear differential equations with a small parameter, *Uspekhi mat. nauk,* 1957, V.12, N 15, 3–122.

[21] Tikhonov A. I., On dependence of solutions of differential equations on small parameter, *Mat. sb.,* 1948, V.28, N 2, 193–204.

[22] Vasyl'eva A. B., On differential equations containing small parameters in derivatives terms, *Mat, sb.,* 1951, V.31, N 3, 587–644.

[23] Lomov S. A., Introduction to the general theory of singular disturbance, Nauka, Moscow, 1981, 398p.

[24] Shkil M. I., Starun I. I., Yakovets V. P., Asymptotic integrating linear systems of differential equations with singularity, Vyshcha shkola, Kyiv, 1991, p.207.