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Symmetry Properties and Reduction of the Generalized Nonlinear System of Two-Phase Liquid Equations

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Let us consider the multidimensional nonlinear system of heat equations

$$\begin{cases} u_0 = f(v)\Delta u; \\ v_0 = \Delta u, \end{cases} \tag{1}$$

where $u = u(x) \in R_1, v = v(x) \in R_1, x = (x_0, \vec{x}) \in R_{1+3}, \Delta$ is the Laplace operator, $f(v)$ is an arbitrary differentiable function.

In this paper the classification of symmetry properties of equations (1) is investigated depending on the function $f(v)$. In the case where the system (1) is invariant with respect to the conformal algebra $AC(3)$, we use the symmetry to construct ansatzes and reduce this system to partial differential equations (PDE).

The following statement takes place.

Theorem. *The basis widest invariance algebra (WIA) of system (1) consists of the operators:*

1) $A_5 = \langle \partial_0, \partial_a, \beta(\vec{x})\partial_u, \quad J_{ab} = x_a\partial_b - x_b\partial_a, \quad D_0 = 2x_0\partial_0 + x_a\partial_a\rangle$, *where* $\beta(\vec{x})$ *is an arbitrary linear function, where* $f(v)$ *is an arbitrary differentiable function;*

2) $A_5,\ D = x_b\partial_b - \dfrac{n-2}{2}u\partial_u - \dfrac{n+2}{2}v\partial_v$, *when* $f(v) = \lambda \exp v$*;*

3) $A_5,\ D_1 = mx_0\partial_0 - (m+1)u\partial_u + (m-1)v\partial_v$, *when* $f(v) = \lambda v^m, m \neq -\dfrac{4}{n+2}$*;*

4) $A_5,\ D, |K_a = 2x_aD - \vec{x}^2\partial_a$, *when* $f(v) = \lambda v^{\frac{4}{n+2}}$*.*

At $n = 3$ and $f(v) = \lambda u^{-4/5}$, system (1) takes the form

$$\begin{cases} u_0 = \lambda u^{-\frac{4}{5}}\Delta u; \\ v_0 = \Delta u. \end{cases} \tag{2}$$

As appears from the above, the WIA of system (2) is the algebra

$$A = \{AC(3), \partial_0, D_0\}.$$

Since the algebra $AC(3)$ is nonlinear with respect to $\vec{x}$, the method of finding invariants that was described in [1] is useless. It is known from [2] that the algebra $AC(3)$ is isomorphous to the algebra $AO(1,4)$ in the passage from the space $\vec{x} \in R_3$ to the space $z = (z_0, \vec{x}, z_4) \in R_{1+4}$ according to the formula

$$\vec{x} = \frac{\vec{z}}{z_4 - z_0}, \tag{3}$$

that realized at the cone

$$z_0^2 - \vec{z}^2 - z_4^2 = 0. \tag{4}$$

The basis elements of the algebra $AO(1,4)$ have the form

$$J_{ab} = z^A \partial_B - z^B \partial_A, \quad A, B = \overline{0,4}.$$

Thus, the algebra A is isomorphous to the algebra

$$\tilde{A} = \{AO(1,4), \partial_0, D_0\}.$$

A connection between the basis operators of the algebras A and $\tilde{A}$ is defined by the formulae

$$\begin{cases} \partial_a &= J_{4a} - J_{0a}; \\ J_{ab} &= J_{ab}; \\ D &= J_{04}; \\ K_a &= J_{4a} + J_{0a}. \end{cases}$$

The algebra $\tilde{A}$ is realized by linear representations and because we can use the method from [1] for finding its invariants. It's necessary to solve the system of differential equations

$$\dot{Z} = A \cdot Z, \tag{5}$$

where the matrix A corresponds to the algebra AO(1,4). Integrating system (5), we find invariants of the algebra AO(1,4). Then according to formulae (3), (4), we obtain the algebra AC(3) and construct ansatzes that reduce system (2) to PDE.

	Invariants	Ansatzes	Reduced Equations
1.1	$\begin{cases} \omega_1 = \vec{a}\vec{x} - x_0 \\ \omega_2 = \vec{b}\vec{x} \\ \omega_3 = \vec{c}\vec{x} \end{cases}$	$\begin{cases} v = \varphi^0(\omega) \\ u = \varphi^1(\omega) \end{cases}$	$\begin{cases} \varphi_1^1 = \lambda(\varphi^0)^{-4/5}\varphi_1^0 \\ -\varphi_1^0 = \Delta\varphi^1 \end{cases}$
1.2	$\begin{cases} \omega_1 = \vec{a}\vec{x} - lnx_0 \\ \omega_2 = \vec{b}\vec{x} \\ \omega_3 = \vec{c}\vec{x} \end{cases}$	$\begin{cases} v = x_0^{5/4}\varphi^0(\omega) \\ u = x_0^{1/4}\varphi^1(\omega) \end{cases}$	$\begin{cases} \frac{\varphi^1}{4} - \varphi_1^1 = \lambda(\varphi^0)^{-4/5}\times \\ \times\left[\frac{5}{4}\varphi^0 - \varphi_1^0\right] \\ \frac{5}{4}\varphi^0 - \varphi_1^0 = \Delta\varphi^1 \end{cases}$
1.3	$\begin{cases} \omega_1 = x_0 \\ \omega_2 = \vec{b}\vec{x} \\ \omega_3 = \vec{c}\vec{x} \end{cases}$	$\begin{cases} v = \varphi^0(\omega) \\ u = \varphi^1(\omega) \end{cases}$	$\begin{cases} \varphi_1^1 = \lambda(\varphi^0)^{-4/5}\varphi_1^0 \\ \varphi_1^0 = \varphi_{22}^1 + \varphi_{33}^1 \end{cases}$
2.1	$\begin{cases} \omega_1 = \dfrac{2}{\vec{x}^2+1}\times \\ (\vec{a}\vec{x}\cos t - \sin t) + \sin t \\ \omega_2 = \dfrac{2}{\vec{x}^2+1}\times \\ (\vec{a}\vec{x}\sin t + \cos t) + \cos t \\ \omega_3 = \dfrac{2}{\vec{x}^2+1}\times \\ \left(\vec{b}\vec{x}\cos mt + \vec{c}\vec{x}\sin mt\right) \\ t = x_0 \end{cases}$	$\begin{cases} v = (\vec{x}^2+ \\ +1)^{-\frac{5}{2}}\varphi^0(\omega) \\ u = (\vec{x}^2+ \\ +1)^{-\frac{1}{2}}\varphi^1(\omega) \end{cases}$	$\begin{cases} & \omega_1\varphi_2^1 - \omega_2\varphi_1^1 + m\varphi_3^1\times \\ \times & (1-\omega_a^2)^{1/2} = \lambda(\varphi^0)^{-4/5}\times \\ \times & [\omega_1\varphi_2^0 - \omega_2\varphi_1^2+ \\ + & m\varphi_3^0(1-\omega_a^2)^{1/2}] \\ & \omega_1\varphi_2^0 - \omega_2\varphi_1^0 + m\varphi_3^0\times \\ \times & (1-\omega_a^2)^{1/2} = -3\varphi^1- \\ - & 24(\omega_1\varphi_1^1 + \omega_2\varphi_2^1)- \\ - & 12\omega_3\varphi_3^1 + 4(1-\omega_a^2)\varphi_{aa}^1- \\ - & 8(\omega_1\omega_2\varphi_{12} + \omega_1\omega_3\varphi_{13}+ \\ + & \omega_2\omega_3\varphi_{23} \end{cases}$

	Invariants	Ansatzes	Reduced Equations
2.2	$\begin{cases} \omega_1 = \dfrac{2}{\vec{x}^2+1}(\vec{a}\vec{x}\cos t - \\ - \sin t) + + \sin t \\ \omega_2 = \dfrac{2}{\vec{x}^2+1}(\vec{a}\vec{x}\sin t + \\ + \cos t) - \cos t \\ \omega_3 = \dfrac{2}{\vec{x}^2+1}(\vec{b}\vec{x}\cos mt + \\ + \vec{c}\vec{x}\sin mt) \\ t = ln x_0 \end{cases}$	$\begin{cases} v = (\vec{x}^2+1)^{-\frac{5}{2}} \times \\ \times x_0^{5/4}\varphi^0(\omega) \\ u = (\vec{x}^2+1)^{-\frac{1}{2}} \times \\ \times x_0^{1/4}\varphi^1(\omega) \end{cases}$	$\begin{cases} \omega_1\varphi_2^1 - \omega_2\varphi_1^1 + m\varphi_3^1 \times \\ \times(1-\omega_a^2)^{1/2} = \lambda(\varphi^0)^{-4/5} \times \\ \times[\omega_1\varphi_2^0 - \omega_2\varphi_1^0 + m\varphi_3^0 \times \\ \times(1-\omega_a^2)^{1/2} + \frac{5}{4}\varphi^0] - \\ -\frac{1}{4}\varphi^1 \\ \omega_1\varphi_2^0 - \omega_2\varphi_1^0 + m\varphi_3^0 \times \\ \times(1-\omega_a^2)^{1/2} = -3\varphi^1 - \\ -24(\omega_1\varphi_1^1 + \omega_2\varphi_2^1) - \\ -12\omega_3\varphi_3^1 + 4(1-\omega_a^2)\varphi_{aa}^1 - \\ -8(\omega_1\omega_2\varphi_{12} + \omega_1\omega_3\varphi_{13} + \\ +\omega_2\omega_3\varphi_{23}) - \frac{5}{4}\varphi^0 \end{cases}$
2.3	$\begin{cases} \omega_1 = \dfrac{2}{\vec{x}^2+1}(\vec{a}\vec{x}\cos t - \\ - \sin t) + \sin t \\ \omega_2 = \dfrac{2}{\vec{x}^2+1}(\vec{a}\vec{x}\sin t + \\ + \cos t) - \cos t \\ \omega_3 = x_0 \\ t = \alpha \arctan \dfrac{\vec{c}\vec{x}}{\vec{b}\vec{x}} \end{cases}$	$\begin{cases} v = (\vec{x}^2+1)^{-\frac{5}{2}} \times \\ \times \varphi^0(\omega) \\ u = (\vec{x}^2+1)^{-\frac{1}{2}} \times \\ \times \varphi^1(\omega) \end{cases}$	$\begin{cases} \varphi_3^1 = \lambda(\varphi^0)^{-4/5}\varphi_3^0 \\ \varphi_3^0 = 3\varphi^1 - 4(\omega_1\varphi_1^1 + \omega_2\varphi_2^1) \times \\ \times\left(3 + \dfrac{\alpha^2}{1-\omega_1^2-\omega_2^2}\right) + \\ +4\varphi_{11}^1\left(1-\omega_1^2 + \dfrac{\alpha^2\omega_2^2}{1-\omega_1^2-\omega_2^2}\right) \\ +4\varphi_{22}^1\left(1-\omega_2^2 + \dfrac{\alpha^2\omega_1^2}{1-\omega_1^2-\omega_2^2}\right) \\ +2\varphi_{12}^1\omega_1\omega_2\left(4 + \dfrac{\alpha^2}{1-\omega_1^2-\omega_2^2}\right) \end{cases}$
3.1	$\begin{cases} \omega_1 = e^{x_0\alpha}\left(\frac{1}{2} - \dfrac{\vec{a}\vec{x}+1}{1-\vec{x}^2}\right) \\ \omega_2 = \dfrac{\vec{b}\vec{x}}{1-\vec{x}^2} \\ \omega_3 = \dfrac{\vec{c}\vec{x}}{1-\vec{x}^2} \end{cases}$	$\begin{cases} v = (1-\vec{x}^2)^{-\frac{5}{2}} \times \\ \times \varphi^0(\omega) \\ u = (1-\vec{x}^2)^{-\frac{1}{2}} \times \\ \times \varphi^1(\omega) \end{cases}$	$\begin{cases} \varphi_1^1 = \lambda(\varphi^0)^{-4/5}\varphi_1^0 \\ \alpha\omega_1\varphi_1^0 = 3\varphi^1 + 12\nabla\varphi^1\vec{\omega} + \\ +\varphi_{22}^1 + \varphi_{33}^1 + 4\omega_a\omega_a\omega_b\varphi_{ab}^1 \end{cases}$
3.2	$\begin{cases} \omega_1 = x_0^\alpha\left(\dfrac{1}{2} - \dfrac{\vec{a}\vec{x}+1}{1-\vec{x}^2}\right) \\ \omega_2 = \dfrac{\vec{b}\vec{x}}{1-\vec{x}^2} \\ \omega_3 = \dfrac{\vec{c}\vec{x}}{1-\vec{x}^2} \end{cases}$	$\begin{cases} v = (1-\vec{x}^2)^{-\frac{5}{2}} \times \\ \times x_0^{5/4}\varphi^0(\omega) \\ u = (1-\vec{x}^2)^{-\frac{1}{2}} \times \\ \times x_0^{1/4}\varphi^1(\omega) \end{cases}$	$\begin{cases} \dfrac{\varphi^1}{4} + \alpha\omega_1\varphi_1^1 = \\ = \lambda(\varphi^0)^{-\frac{4}{5}}[-\frac{5}{4}\varphi^0 + \alpha\omega_1\varphi_1^0] \\ -\frac{5}{4}\varphi^0 + \alpha\omega_1\varphi_1^0 = 3\varphi^1 + \\ +12\nabla\varphi^1\vec{\omega} + \varphi_{22}^1 + \varphi_{33}^1 + \\ +4\omega_a\omega_b\varphi_{ab}^1 \end{cases}$
3.3	$\begin{cases} \omega_1 = x_0 \\ \omega_2 = \dfrac{\vec{b}\vec{x}}{1-\vec{x}^2} \\ \omega_3 = \dfrac{\vec{c}\vec{x}}{1-\vec{x}^2} \end{cases}$	$\begin{cases} v = (1-\vec{x}^2)^{-\frac{5}{2}} \times \\ \times \varphi^0(\omega) \\ u = (1-\vec{x}^2)^{-\frac{1}{2}} \times \\ \times \varphi^1(\omega) \end{cases}$	$\begin{cases} \varphi_1^1 = \lambda(\varphi^0)^{-4/5}\varphi_1^0 \\ \varphi_1^0 = 3\varphi^1 + 12(\omega_2\varphi_2^1 + \omega_3\varphi_3^1) + \\ +4\omega_2\omega_3\varphi_{23}^1 + (4\omega_2^2+1)\varphi_{22}^1 + \\ +(4\omega_3^2+1)\varphi_{33}^1 \end{cases}$
4.1	$\begin{cases} \omega_1 = \vec{a}\vec{x} \\ \omega_2 = \vec{b}\vec{x}^2 + \vec{c}\vec{x}^2 \\ \omega_3 = \arctan\dfrac{\vec{b}\vec{x}}{\vec{c}\vec{x}} - \alpha x_0 \end{cases}$	$\begin{cases} v = \varphi^0(\omega) \\ u = \varphi^1(\omega) \end{cases}$	$\begin{cases} \varphi_1^1 = \lambda(\varphi^0)^{-4/5}\varphi_3^0 - \\ -\alpha\varphi_3^0 = 4\varphi_2^1 + \varphi_{11}^1 + \\ +4\omega_2\varphi_{22}^1 + \dfrac{1}{\omega_2}\varphi_{33}^1 \end{cases}$
4.2	$\begin{cases} \omega_1 = \vec{a}\vec{x} \\ \omega_2 = \vec{b}\vec{x}^2 + \vec{c}\vec{x}^2 \\ \omega_3 = \arctan\dfrac{\vec{b}\vec{x}}{\vec{c}\vec{x}} - ln x_0 \end{cases}$	$\begin{cases} v = x_0^{5/4} \times \\ \times \varphi^0(\omega) \\ u = x_0^{1/4} \times \\ \times \varphi^1(\omega) \end{cases}$	$\begin{cases} \frac{\varphi^1}{4} - \alpha\varphi_2^1 = \lambda(\varphi^0)^{-1/5} \times \\ \times\left[\frac{5}{4}\varphi^0 - \alpha\varphi_3^0\right] \\ \frac{5}{4}\varphi^0 - \alpha\varphi_3^0 = 4\varphi_2^1 + \varphi_{11}^1 + \\ +4\omega_2\varphi_{22}^1 + \dfrac{1}{\omega_2}\varphi_{33}^1 \end{cases}$

	Invariants	Ansatzes	Reduced Equations
4.3	$\begin{cases} \omega_1 = \vec{a}\vec{x} \\ \omega_2 = \vec{b}\vec{x}^2 + \vec{c}\vec{x}^2 \\ \omega_3 = x_0 \end{cases}$	$\begin{cases} v = \varphi^0(\omega) \\ u = \varphi^1(\omega) \end{cases}$	$\begin{cases} \varphi^1_3 = \lambda(\varphi^0)^{-4/5}\varphi^0_3 \\ \varphi^0_3 = 4\varphi^1_2 + \varphi^1_{11} + 4\omega_2\varphi^1_{22} \end{cases}$
5.1	$\begin{cases} \omega_1 = e^{t\alpha}\left(\dfrac{1}{2} - \dfrac{\vec{a}\vec{x}+1}{1-\vec{x}^2}\right) \\ \omega_2 = \dfrac{1}{1-\vec{x}^2}(\vec{b}\vec{x}\cos t + \\ +\vec{c}\vec{x}\sin t) \\ \omega_3 = \dfrac{1}{1-\vec{x}^2}(\vec{b}\vec{x}\sin t + \\ +\vec{c}\vec{x}\cos t) \\ t = x_0 \end{cases}$	$\begin{cases} v = (1-\vec{x}^2)^{-\frac{5}{2}}\times \\ \times\varphi^0(\omega) \\ u = (1-\vec{x}^2)^{-\frac{1}{2}}\times \\ \times\varphi^1(\omega) \end{cases}$	$\begin{cases} \alpha\omega_1\varphi^1_1 - \omega_3\varphi^1_2 + \omega_2\varphi^1_3 = \\ = \lambda(\varphi^0)^{-4/5}(\alpha\omega_1\varphi^0_1 - \\ -\omega_3\varphi^0_2 + \omega_2\varphi^0_3) \\ \alpha\omega_1\varphi^0_1 - \omega_3\varphi^0_2 + \omega_2\varphi^0_3 = \\ = 3\varphi^1 + 12\nabla\varphi^1\vec{\omega} + \\ +8(\omega_1\omega_2\varphi_{12} + \omega_1\omega_3\varphi_{13} + \\ +\omega_2\omega_3\varphi_{23}) + 4\omega_a^2\varphi^1_{aa} + \\ +\varphi^1_{22} + \varphi^1_{33} \end{cases}$
5.2	$\begin{cases} \omega_1 = e^{t\alpha}\left(\dfrac{1}{2} - \dfrac{\vec{a}\vec{x}+1}{1-\vec{x}^2}\right) \\ \omega_2 = \dfrac{1}{1-\vec{x}^2}(\vec{b}\vec{x}\cos t + \\ +\vec{c}\vec{x}\sin t) \\ \omega_3 = \dfrac{1}{1-\vec{x}^2}(\vec{b}\vec{x}\sin t + \\ +\vec{c}\vec{x}\cos t) \\ t = lnx_0 \end{cases}$	$\begin{cases} v = (1-\vec{x}^2)^{-\frac{5}{2}}\times \\ \times x_0^{5/4}\varphi^0(\omega) \\ u = (1-\vec{x}^2)^{-\frac{1}{2}}\times \\ \times x_0^{1/4}\varphi^0(\omega) \end{cases}$	$\begin{cases} \alpha\omega_1\varphi^1_1 - \omega_3\varphi^1_2 + \omega_2\varphi^1_3 = \\ = \lambda(\varphi^0)^{-4/5}(\alpha\omega_1\varphi^0_1 - \\ -\omega_3\varphi^0_2 + \omega_2\varphi^0_3 + \frac{5}{4}\varphi^0) - \\ -\frac{1}{4}\varphi^1 \\ \alpha\omega_1\varphi^0_1 - \omega_3\varphi^0_2 + \omega_2\varphi^0_3 = \\ = 3\varphi^1 + 12\nabla\varphi^1\vec{\omega} + \\ +8(\omega_1\omega_2\varphi_{12} + \omega_1\omega_3\varphi_{13} + \\ +\omega_2\omega_3\varphi_{23}) + 4\omega_a^2\varphi^1_{aa} + \\ +\varphi^1_{22} + \varphi^1_{33} - \frac{5}{4}\varphi^0 \end{cases}$
5.3	$\begin{cases} \omega_1 = \left(\dfrac{1}{2} - \dfrac{\vec{a}\vec{x}+1}{1-\vec{x}^2}\right)\times \\ \times\exp\left(\alpha\arctan\dfrac{\vec{c}\vec{x}}{\vec{b}\vec{x}}\right) \\ \omega_2 = \dfrac{\vec{b}\vec{x}+\vec{c}\vec{x}}{1-\vec{x}^2} \\ \omega_3 = x_0 \end{cases}$	$\begin{cases} v = (1-\vec{x}^2)^{-\frac{5}{2}}\times \\ \times\varphi^0(\omega) \\ u = (1-\vec{x}^2)^{-\frac{1}{2}}\times \\ \times\varphi^1(\omega) \end{cases}$	$\begin{cases} \varphi^1_3 = \lambda(\varphi^0)^{-4/5}\varphi^0_3 \\ \varphi^0_3 = 3\varphi^1 + \left(4\omega_1^2 + \frac{\alpha^2\omega_1^2}{\omega_2^2}\right)\times \\ \times\varphi^1_{11} + (4\omega_2^2+1)\varphi^1_{22} + \\ +8\omega_1\omega_2\varphi^1_{12} + \\ +\left(12\omega_1 + \dfrac{\alpha^2\omega_1}{\omega_2^2}\right)\varphi^1_1 + \\ +\left(12\omega_2 + \frac{1}{\omega_2}\right)\varphi^1_2 \end{cases}$

Below, some exact solutions of system (2) are given

$$\begin{cases} v = \left\{\dfrac{\lambda x_0}{(\vec{b}\vec{x})^2 + (\vec{c}\vec{x})^2}\right\}^{5/4}; \\ u = 5\lambda\left\{\dfrac{\lambda x_0}{(\vec{b}\vec{x})^2 + (\vec{c}\vec{x})^2}\right\}^{1/4}, \end{cases} \qquad \begin{cases} v = x_0^{5/4}y^5; \\ u = x_0^{1/4}5\lambda y, \end{cases}$$

where $y = y(\omega_1)$ is a solution of the equation

$$\lambda y'' + y^4 y' - \frac{1}{4}y^5 = 0,$$

$$\begin{cases} v = \left\{\dfrac{3\lambda x_0}{(\vec{b}\vec{x})^2}\right\}^{5/4}; \\ u = 5\lambda\left\{\dfrac{3\lambda x_0}{(\vec{b}\vec{x})^2}\right\}^{1/4}, \end{cases} \qquad \begin{cases} v = \left\{\frac{-\lambda x_0}{\vec{x}^2}\right\}^{5/4}; \\ u = 5\lambda\left\{\frac{-\lambda x_0}{\vec{x}^2}\right\}^{1/4}, \end{cases}$$

$$\begin{cases} v = \left\{ \dfrac{12\lambda x_0}{(1-\vec{x}^2)^2} \right\}^{5/4}; \\ u = 5\left\{ \dfrac{\lambda^5 x_0}{12(1-\vec{x}^2)^2} \right\}^{1/4}, \end{cases} \qquad \begin{cases} v = \left\{ \dfrac{-12\lambda x_0}{(1+\vec{x}^2)^2} \right\}^{5/4}; \\ u = 5\left\{ \dfrac{-\lambda^5 x_0}{12(1+\vec{x}^2)^2} \right\}^{1/4}. \end{cases}$$

References

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[2] Fushchych W.I., Barannik A.F., Barannik L.F., Subgroup Analysis of the Galilean, Poincaré Groups and Reduction of Nonlinear Equations, Kyiv, Naukova Dumka, 1991, 304p.