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On Symmetry Reduction of Nonlinear Generalization of the Heat Equation

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Abstract

Reductions and classes of new exact solutions are constructed for a class of Galilei-invariant heat equations.

It is well-known that the n-dimensional linear heat equation

$$ku_t = u_{11} + \ldots + u_{nn} \tag{1}$$

where $u_t = \dfrac{\partial u}{\partial t}, u_{ij} = \dfrac{\partial^2 u}{\partial x_i \partial x_j}$, is invariant under the extended complete Galilei algebra $A\tilde{G}_2(1,n)$. Unfortunately, the equation (1) cannot describe a great number of real processes of heat and mass transfer. The known nonlinear generalization of the equation (1)

$$u_t + \nabla(F(u)\nabla u) = 0 \tag{2}$$

is invariant under the Galilei algebra only if $F(u) = \text{const}$. Galilei-invariant nonlinear generalizations of the equation (1) were described in the paper [1].

Let formulate the necessary results. Consider the equation of the second order

$$u_t + F(t, \overline{x}, u, \underset{1}{u}, \underset{2}{u}) = 0, \tag{3}$$

where $\underset{s}{u}$ is the set of s-th order partial derivatives of u with respect to the space variables $x_1, x_2, \ldots, x_n$ $(s = 1, 2)$.

The equation (3) is invariant under the extended classical Galilei algebra $A\tilde{G}(1,n)$ iff it is of the form

$$u_t + \frac{1}{2m}(\nabla u)^2 + \Phi(<1>;<2>;\ldots;<n>) = 0, \tag{4}$$

where Φ is an arbitrary smooth function, $m = \text{const}$,

$$< 1 >= u_{11} + u_{22} + \ldots + u_{nn},$$

$$< 2 >= \begin{vmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{vmatrix} + \begin{vmatrix} u_{11} & u_{13} \\ u_{31} & u_{33} \end{vmatrix} + \ldots + \begin{vmatrix} u_{n-1n-1} & u_{n-1n} \\ u_{nn-1} & u_{nn} \end{vmatrix},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$< n >= \begin{vmatrix} u_{11} & u_{12} & \ldots & u_{1n} \\ u_{21} & u_{22} & \ldots & u_{2n} \\ \vdots & \vdots & & \vdots \\ u_{n1} & u_{n2} & \ldots & u_{nn} \end{vmatrix},$$

i.e., $< k >$ is the sum of k-th order minors of the main diagonal of the matrix (u_{ij}).

The basis of the algebra $A\tilde{G}(1, n)$ is formed by the following vector fields

$$P_a = \partial_a, \quad G_a = t\partial_a + mx_a\partial_u, \quad T = \partial_t,$$

$$J_{ab} = x_a\partial_b - x_b\partial_a, \quad M = m\partial_u,$$

where $\partial_t = \dfrac{\partial}{\partial t}, \quad \partial_u = \dfrac{\partial}{\partial u}, \quad \partial_a = \dfrac{\partial}{\partial x_a} \quad (a < b; a, b = 1, \ldots, n).$

If L is a subalgebra of the rank r of the algebra $A\tilde{G}(1, n)$, $s = n+2-r$ and $\omega_1(t, \overline{x}), \ldots, \omega_{s-1}(t, \overline{x}), \omega_s(t, \overline{x}, u)$ are the functionally independent invariants of L, then the ansatz $\omega_s = \varphi(\omega_1, \ldots, \omega_{s-1})$ reduces the equation (4) to a differential equation containing only φ, ω_i, and derivatives $\dfrac{\partial\varphi}{\partial\omega_i}, \dfrac{\partial^2\varphi}{\partial\omega_i\partial\omega_i}$ where $i, j, = 1, \ldots, s-1$ (see [2]). Such a reduction is called a symmetry reduction.

In the present paper, the symmetry reduction of the equation (4) to ordinary differential equations is carried out.

It is not difficult to convince of that subalgebras of the rank n of the algebra invariance of the equation (4) considered with respect to $\tilde{G}(1, n)$-conjugation will be the same as for the algebra of invariance of the n-dimensional nonlinear Schrödinger equation

$$2mi\,\frac{\partial\psi}{\partial t} = \frac{\partial^2\psi}{\partial x_1^2} + \ldots + \frac{\partial^2\psi}{\partial x_n^2} + \psi F(|\psi|) = 0,$$

where F is an arbitrary smooth function. It allows us to use the results of the paper [3].

As in [3], in the present paper we confine ourselves by consideration of such subalgebras of the rank n which do not contain operator M.

Let $AO[p, q] =< J_{ab}; a, b = p, \ldots, q >$;

$\Phi(d_0, d_1, \gamma_1) =< G_{d_0} + \gamma_1 P_{d_0}, \ldots, G_{d_0} + \gamma_1 P_{d_1} > +)AO[d_0, d_1]$;

$AE(n-k) =< P_{k+1}, \ldots, P_n > +)AO[k+1, n] \ (0 \le k \le n-1)$;

$AE(n-n) = AE(0) = 0$;

$AE_1(n-k) =< G_{k+1}, \ldots, G_n > +)AO[k+1, n] \ (0 \le k \le n-1)$;

$AE_1(n-n) = AE_1(0) = 0$.

Let $d_1, \ldots, d_p$ be natural numbers which satisfy the condition $1 = d_0 < d_1 < \ldots < d_p \le n$. With respect to $\tilde{G}(1, n)$-conjugation, the algebra $A\tilde{G}(1, n)$ contains 6 maximal subalgebras of the rank n. For each of these algebras we show a corresponding ansatz and reduced equation.

1) $AE(n): \quad u = \varphi(t), \quad \dot{\varphi} + \Phi(0; 0; \ldots; 0) = 0.$

2) $\Phi(1, d_1, \gamma_1) \bigoplus \ldots \bigoplus \Phi(d_{p-1}+1, d_p, \gamma_p) \bigoplus AE(n-k)$ $(d_p = m; 1 \le k \le n)$:

$$u = \frac{m}{2} \sum_{j=1}^{p} \frac{x^2_{d_{j-1}} + \ldots + x^2_{d_j}}{t - \gamma_j} \ + \ \varphi(t), \quad \dot{\varphi} + \Phi(m\sigma_1; m^2\sigma_2; \ldots, m^k\sigma_k; 0; \ldots; 0) = 0,$$

where

$\sigma_1 = y_1 + y_2 + \ldots + y_k,$

$\sigma_2 = y_1y_2 + y_1y_3 + \ldots + y_{k-1}y_k,$

. . . .

$\sigma_k = y_1y_2 \ldots y_k$

are the elementary symmetrical polynomials and $y_1 = \ldots = y_{d_1} = \dfrac{1}{\omega - \gamma_1}$,

$y_{d_1+1} = \ldots = y_{d_2} = \dfrac{1}{\omega - \gamma_2}, \ldots, \ y_{d_{p-1}+1} = \ldots = y_{d_p} = \dfrac{1}{\omega - \gamma_p}.$

3) $< T + \alpha M, J_{12} + \beta M > \bigoplus AE(n-2)$ $(\alpha, \beta \in R)$:

$$u = \alpha mt + \beta \arctan \frac{x_1}{x_2} + \varphi(x_1^2 + x_2^2),$$

$$\alpha m + \frac{1}{2m}(\beta^2\omega^{-1} + 4\omega\dot{\varphi}^2) + \Phi(4\dot{\varphi} + 4\omega\ddot{\varphi}; 4\dot{\varphi}^2 + 8\omega\dot{\varphi}\ddot{\varphi} - \beta^2\omega^{-2}; 0; \ldots; 0) = 0.$$

4) $< T + \alpha M > \bigoplus AE(n-1)$ $(\alpha \in R)$:

$$u = \alpha mt + \varphi(x_1), \quad \alpha m + \frac{1}{2m}\dot{\varphi}^2 + \Phi(\ddot{\varphi}; 0; \ldots; 0) = 0.$$

5) $< T + \alpha G_1 > \bigoplus AE(n-1)$ $(\alpha > 0)$:

$$u = \alpha mtx_1 - \frac{1}{3}\alpha^2 mt^3 + \varphi(\alpha t^2 - 2x_1), \quad -\frac{\alpha m}{2}\omega + \frac{2}{m}\dot{\varphi}^2 + \Phi(4\ddot{\varphi}; 0; \ldots; 0) = 0.$$

6) $< T + \alpha M > \bigoplus AO[1, k] \bigoplus AE(n-k)$ $(\alpha \in R; 3 \le k \le n)$:

$$u = \alpha mt + \varphi\Big(\sum_{i=1}^{k} x_i^2\Big), \quad \alpha m + \frac{2}{m}\omega\dot{\varphi}^2 + \Phi(y_1; \ldots; y_k; 0; \ldots; 0) = 0,$$

where $y_p = \dfrac{2^p(k-1)!}{(k-p)!p!}(\dot{\varphi})^{p-1}(k\dot{\varphi} + 2p\omega\ddot{\varphi})(p = 1, \ldots, k).$

References

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[2] Ovsyannikov L.V., Group Analysis of Differential Equations, 1982, 400p.

[3] Barannik A.F., Marchenko V.A., Fushchych W.I., On reduction and exact solutions of the nonlinear multidimensional Schrödinger equations, *Teoret. Matemat. Fizika,* 1991, V.87, N 2, 220–234.