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On the ten classes of scale-invariant nonlinear wave equations for vector fields

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Abstract

We describe all systems of three equations of the form $\Box u_j = F_j(u_1, u_2, u_3), j = \overline{1, 3}$ invariant under the extended Poincaré group. As a result, we have obtained ten classes of $\tilde{P}(1, 3)$-invariant nonlinear partial differential equations for real vector fields.

It is well known that the maximal symmetry group admitted by the nonlinear wave equation

$$\Box u \equiv u_{x_0 x_0} - \triangle_3 u = F(u) \tag{1}$$

with an arbitrary smooth function $F(u)$ is the 10-parameter Poincaré group $P(1, 3)$ having the following generators:

$$P_\mu = \partial_\mu, \qquad J_{\mu\nu} = g_{\mu\alpha} x_\alpha \partial_\nu - g_{\nu\alpha} x_\alpha \partial_\mu, \tag{2}$$

where $\partial_\mu = \partial/\partial_{x_\mu}$ $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, $\mu, \nu, \alpha = 0, 1, 2, 3$. Hereafter, the summation over the repeated indices from 0 to 3 is understood.

As established in [3], equation (1) admits the wider symmetry group only in two cases:

$$1. \quad F(u) = \lambda u^k, \qquad k \neq 1, \tag{3}$$
$$2. \quad F(u) = \lambda e^{ku}, \qquad k \neq 0, \tag{4}$$

where λ, k are arbitrary constants.

Equations (1) with nonlinearities (3), (4) admit the one-parameter groups of scale transformations $D(1)$ having the following generators:

$$\begin{aligned} &1. \quad D = x_\mu \partial_\mu + \frac{2}{1-k} u \partial_u, \\ &2. \quad D = x_\mu \partial_\mu - \frac{2}{k} \partial_u. \end{aligned} \tag{5}$$

The group with generators (2), (5) is called the extended Poincaré group $\widetilde{P}(1, 3)$ [1].

Let us note that in [4] two classes of $\widetilde{P}(1, 3)$-invariant equations of the form

$$\Box u = F(u, u^*) \tag{6}$$

were constructed.

In the present paper we give a complete solution of the symmetry classification of systems of three real equations

$$\Box u_j = F_j(u_1, u_2 u_3), \quad j = 1,2,3 \tag{7}$$

admitting the extended Poincaré group $\widetilde{P}(1,3)$ and the conformal group $C(1,3)$.

Before formulating the principal assertions we make a remark. As a direct check shows, a class of equations (7) is invariant under the linear transformations of depended variables

$$u_j \to u'_j = \sum_{k=1}^{3} \alpha_{jk} u_k + \beta_j, \quad j = 1,2,3, \tag{8}$$

where $\alpha_{jk}, \beta_j,\ j = 1,2,3$ are arbitrary constants and $\det\|\alpha_{jk}\| \neq 0$.

That is why we carried out symmetry classification of equations (7) up to the equivalence transformations (8).

Theorem 1. *System of partial differential equations (PDE) (7) is invariant under the extended Poincaré group $\widetilde{P}(1,3)$ iff it is equivalent to one of the following systems (for all cases $F_j = F_j(\omega_1, \omega_2), \quad j = 1,2,3$):*

$$\begin{aligned}
1.\quad \Box u_1 &= F_1 u_1^{\frac{\lambda_1-2}{\lambda_1}}, \\
\Box u_2 &= F_2 u_2^{\frac{\lambda_2-2}{\lambda_2}}, \\
\Box u_3 &= F_3 u_3^{\frac{\lambda_3-2}{\lambda_3}}, \\
\omega_1 &= \frac{u_1^{\lambda_2}}{u_2^{\lambda_1}}; \quad \omega_2 = \frac{u_1^{\lambda_3}}{u_3^{\lambda_1}}; \quad ;
\end{aligned}$$

$$\begin{aligned}
2.\quad \Box u_1 &= F_1 \exp\Big(-\frac{2}{b}u_1\Big), \\
\Box u_2 &= F_2 \exp\Big\{(\lambda_2 - 2)\frac{u_1}{b}\Big\}, \\
\Box u_3 &= F_3 \exp\Big\{(\lambda_3 - 2)\frac{u_1}{b}\Big\}, \\
\omega_1 &= \lambda_2 u_1 - b\ln u_2, \quad \omega_2 = \lambda_3 u_1 - b\ln u_3;
\end{aligned}$$

$$\begin{aligned}
3.\quad \Box u_1 &= \Big\{F_1 + \frac{u_1}{u_2}F_2\Big\} \exp\Big\{(\lambda_1 - 2)\frac{u_1}{u_2}\Big\}, \\
\Box u_2 &= F_2 \exp\Big\{(\lambda_1 - 2)\frac{u_1}{u_2}\Big\}, \\
\Box u_3 &= F_3 \exp\Big\{(\lambda_2 - 2)\frac{u_1}{u_2}\Big\}, \\
\omega_1 &= \frac{\exp\Big(\frac{u_1}{u_2}\lambda_2\Big)}{u_3}, \quad \omega_2 = \frac{\exp\Big(\frac{u_1}{u_2}\lambda_1\Big)}{u_2};
\end{aligned}$$

$$\begin{aligned}
4.\quad \Box u_1 &= (F_1 + F_2 u_2)\exp\Big(-\frac{2}{b}u_2\Big),\\
\Box u_2 &= bF_2\exp\Big(-\frac{2}{b}u_2\Big),\\
\Box u_3 &= F_3\exp\Big\{(\lambda-2)\frac{u_2}{b}\Big\},\\
\omega_1 &= 2bu_1 - u_2^2, \quad \omega_2 = \lambda u_2 - b\ln u_3;
\end{aligned}$$

$$\begin{aligned}
5.\quad \Box u_1 &= (F_1 + F_2 u_3)\exp\Big\{(\lambda-2)\frac{u_3}{b}\Big\},\\
\Box u_2 &= bF_2\exp\Big\{(\lambda-2)\frac{u_3}{b}\Big\},\\
\Box u_3 &= F_3\exp\Big(-\frac{2}{b}u_3\Big),\\
\omega_1 &= b\ln u_2 - \lambda u_3, \quad \omega_2 = b\frac{u_1}{u_2} - u_3;
\end{aligned} \tag{9}$$

$$\begin{aligned}
6.\quad \Box u_1 &= \Big(F_1 + F_2\frac{u_2}{u_3} + F_3\frac{u_1}{u_3}\Big)\exp\Big\{(\lambda-2)\frac{u_2}{u_3}\Big\},\\
\Box u_2 &= \Big(F_2 + F_3\frac{u_2}{u_3}\Big)\exp\Big\{(\lambda-2)\frac{u_2}{u_3}\Big\},\\
\Box u_3 &= F_3\exp\Big\{(\lambda-2)\frac{u_2}{u_3}\Big\},\\
\omega_1 &= \lambda\frac{u_2}{u_3} - \ln u_3, \quad \omega_2 = 2\frac{u_1}{u_3} - \Big(\frac{u_2}{u_3}\Big)^2;
\end{aligned}$$

$$\begin{aligned}
7.\quad \Box u_1 &= \Big(F_1 + F_2\omega_0 + F_3\omega_0^2\Big)\ \exp(-2\omega_0),\\
\Box u_2 &= (F_2 + 2F_3\omega_0)\ \exp(-2\omega_0),\\
\Box u_3 &= 2F_3\exp(-2\omega_0),\\
\omega_1 &= 2u_2 - \frac{u_3^2}{b}, \quad \omega_2 = u_1 - \frac{u_2u_3}{b} + \frac{u_3^3}{3b^2}, \quad \omega_0 = \frac{u_3}{b};
\end{aligned}$$

$$\begin{aligned}
8.\quad \Box u_1 &= (u_1^2 + u_2^2)^{-\frac{1}{2}}(F_1u_2 + F_2u_1)\ \exp\Big(\frac{a-2}{b}\operatorname{arctg}\frac{u_1}{u_2}\Big),\\
\Box u_2 &= (u_1^2 + u_2^2)^{-\frac{1}{2}}(F_2u_2 - F_1u_1)\ \exp\Big(\frac{a-2}{b}\operatorname{arctg}\frac{u_1}{u_2}\Big),\\
\Box u_3 &= F_3\exp\Big(\frac{\lambda-2}{b}\operatorname{arctg}\frac{u_1}{u_2}\Big),\\
\omega_1 &= \frac{(u_1^2+u_2^2)^\lambda}{u_3^{2a}}, \quad \omega_2 = \frac{\exp\Big(\dfrac{\lambda}{b}\operatorname{arctg}\dfrac{u_1}{u_2}\Big\}}{u_3};
\end{aligned}$$

$$
\begin{aligned}
9.\quad \Box u_1 &= \Big(F_1\cos\Big(\frac{b}{c}u_3\Big)+F_2\sin\Big(\frac{b}{c}u_3\Big)\Big)\exp\Big(\frac{a-2}{c}u_3\Big),\\
\Box u_2 &= \Big(F_2\cos\Big(\frac{b}{c}u_3\Big)-F_1\sin\Big(\frac{b}{c}u_3\Big)\Big)\exp\Big(\frac{a-2}{c}u_3\Big),\\
\Box u_3 &= F_3\exp\Big(-\frac{2}{c}u_3\Big),\\
\omega_1 &= \ln(u_1^2+u_2^2)-2a\frac{u_3}{c},\quad \omega_2=\operatorname{arctg}\frac{u_1}{u_2}-b\frac{u_3}{c};
\end{aligned}
$$

$$
\begin{aligned}
10.\quad \Box u_1 &= 0,\\
\Box u_2 &= 0,\\
\Box u_3 &= 0,
\end{aligned}
$$

where F_1, F_2, F_3 are arbitrary smooth functions, a, b, c are arbitrary constants.

Moreover, basic generators P_μ, $J_{\mu\nu}$ are given by formulae (2) and generators of corresponding groups of scale transformations are given by the following formulae:

$$
\begin{aligned}
&1.\quad D=x_\mu\partial_\mu+\lambda_1u_1\partial_{u_1}+\lambda_2u_2\partial_{u_2}+\lambda_3u_3\partial_{u_3},\quad \lambda_1\neq 0;\\
&2.\quad D=x_\mu\partial_\mu+b\partial_{u_1}+\lambda_2u_2\partial_{u_2}+\lambda_3u_3\partial_{u_3};\\
&3.\quad D=x_\mu\partial_\mu+\lambda_1\Big(u_1\partial_{u_1}+u_2\partial_{u_2}\Big)+u_2\partial_{u_1}+\lambda_2u_3\partial_{u_3};\\
&4.\quad D=x_\mu\partial_\mu+u_2\partial_{u_1}+b\partial_{u_2}+\lambda u_3\partial_{u_3};\\
&5.\quad D=x_\mu\partial_\mu+\lambda(u_1\partial_{u_1}+u_2\partial_{u_2})+u_2\partial_{u_1}+b\partial_{u_3};\\
&6.\quad D=x_\mu\partial_\mu+\lambda(u_1\partial_{u_1}+u_2\partial_{u_2}+u_3\partial_{u_3})+u_2\partial_{u_1}+u_3\partial_{u_2};\\
&7.\quad D=x_\mu\partial_\mu+u_2\partial_{u_1}+u_3\partial_{u_2}+b\partial_{u_3};\\
&8.\quad D=x_\mu\partial_\mu+a(u_1\partial_{u_1}+u_2\partial_{u_2})+b(u_2\partial_{u_1}-u_1\partial_{u_2})+\lambda u_3\partial_{u_3};\\
&9.\quad D=x_\mu\partial_\mu+a(u_1\partial_{u_1}+u_2\partial_{u_2})+b(u_2\partial_{u_1}-u_1\partial_{u_2})+c\partial_{u_3};\\
&10.\quad D=x_\mu\partial_\mu,\ \text{all over}\quad b\neq 0,\quad c\neq 0.
\end{aligned}
\tag{10}
$$

Theorem 2. *System of PDE (7) is invariant under the conformal group $C(1,3)$ iff it is equivalent to the following system:*

$$\Box u_j=u_1^3\widetilde{F}_j\left(\frac{u_1}{u_2},\frac{u_1}{u_3}\right),\qquad j=1,2,3.$$

Proofs of the theorems 1, 2 are carried out with the use of the algorithm of S.Lie (see, e.g. [1, 5, 6]). Here we present the scheme of proof of Theorem 1 only.

Within the framework of Lie's approach we look for symmetry operator for system of PDE (7) in the form

$$X=\xi_\mu(x,u)\partial_\mu+\eta_1(x,u)\partial_{u_1}+\eta_2(x,u)\partial_{u_2}+\eta_3(x,u)\partial_{u_3},\tag{11}$$

where $\xi_\mu(x,u)$, $\eta_j(x,u)$ are some smooth functions.

The necessary and sufficient condition for system of PDE (7) to be invariant under the group having the infinitesimal operator (11) reads

$$\widetilde{X}(\Box u_j + F_j)\Big|_{\substack{\Box u_1 - F_1 = 0\\ \Box u_2 - F_2 = 0\\ \Box u_3 - F_3 = 0}} = 0, \quad j = 1, 2, 3, \tag{12}$$

where $\widetilde{X}$ is the second prolongation of the operator X.

Splitting relations (12) with respect to independent variables, we get the Killing-type system of PDE for ξ_μ, η_k. Integrating it, we have:

$$\begin{aligned} \xi_\mu &= 2x_\mu g_{\alpha\beta} x_\alpha k_\beta - k_\mu g_{\alpha\beta} x_\alpha x_\beta + \\ &\qquad\qquad c_{\mu\alpha} g_{\alpha\beta} x_\beta + d x_\mu + e_\mu, \quad \mu = \overline{0,3}, \\ \eta_k &= \sum_{j=1}^{3} a_{kj} u_j + b_k(x) - 2 g_{\alpha\beta} k_\alpha x_\beta u_k, \quad k = 1, 2, 3, \end{aligned} \tag{13}$$

where k_α, $c_{\mu\nu} = -c_{\mu\nu}$, d, e_μ, a_{kj} are arbitrary constants, $b_k(x)$ are arbitrary functions satisfying the following relations:

$$\begin{aligned} &\sum_{k=1}^{3}\left(\sum_{l=1}^{3} a_{kj} u_l + b_k(x) - 2 g_{\alpha\beta} k_\alpha x_\beta u_k\right) F_{j u_k} + \Box b_j(x) + \\ &\qquad\qquad 2(d + 3 g_{\alpha\beta} k_\alpha x_\beta) F_j - \sum_{l=1}^{3} a_{jl} F_l = 0, \quad j = 1, 2, 3. \end{aligned} \tag{14}$$

From (13), (14) it follows that system of PDE (7) is invariant under the Poincaré group $P(1,3)$ having the generators (2) with arbitrary F_1, F_2. To describe all functions F_1, F_2 such that system (7) admits the extended Poincaré group $\widetilde{P}(1,3)$ one has to solve two problems:

1) to describe all operators D of the form (11), (13) which together with operators (2) satisfy the commutational relations of the Lie algebra of the group $\widetilde{P}(1,3)$ [1]:

$$\begin{aligned} &[P_\alpha, P_\beta] = 0, \quad [P_\alpha, J_{\beta\gamma}] = g_{\alpha\beta} P_\gamma - g_{\alpha\gamma} P_\beta, \\ &[J_{\alpha\beta}, J_{\mu\nu}] = g_{\alpha\nu} J_{\beta\mu} + g_{\beta\mu} J_{\alpha\nu} - g_{\alpha\mu} J_{\beta\nu} - g_{\beta\nu} J_{\alpha\mu}, \\ &[D, J_{\alpha\beta}] = 0, \quad [P_\alpha, D] = P_\alpha, \quad \alpha, \beta, \gamma, \mu, \nu = \overline{0,3}; \end{aligned}$$

2) to solve system of PDE (14) for each operator D obtained.

On solving the first problem we establish that the operator D has the form

$$D = x_\mu \partial_\mu + \sum_{i=1}^{3}\left(\sum_{j=1}^{3} A_{ij} u_j + B_i\right)\partial_{u_i}, \tag{15}$$

where A_{ij}, B_i are arbitrary constants.

As noted above, two operators D and D' connected by the transformation (8) (which does not alter the form of the operators P_μ, $J_{\mu\nu}$), are considered as equivalent. Using this fact, one can simplify essentially the form of operator (15).

Making in (15) the change of variables (8) with $\beta_j = 0$ we have

$$D' = x_\mu \partial_\mu + \sum_{i=1}^{3} \left(\sum_{j=1}^{3} \widetilde{A}_{ij} u'_j + \widetilde{B}_i \right) \partial_{u'_i},$$

where

$$\begin{aligned} &\|\widetilde{A}_{ij}\| = \|\alpha_{ij}\| \, \|A_{ij}\| \, \|\alpha_{ij}\|^{-1}, \\ &\widetilde{B}_i = \sum_{k=1}^{3} \alpha_{ik} B_k, \quad i = 1, 2, 3. \end{aligned} \tag{16}$$

Since an arbitrary (3×3)-matrix can be reduced to the Jordan form by transformation (16) we may assume, without loss of generality, that the matrix $\|\widetilde{A}_{ij}\|$ is in the Jordan form. The further simplification of the form of operator (15) is achieved at the expense of transformation (8) with $\alpha_{ik} = 0$.

As a result, the set of operators (15) is divided into ten equivalence classes, whose representatives are adduced in (10).

Next, integrating corresponding system of PDE (14), we get $\widetilde{P}(1,3)$-invariant systems of equations (9).

Note 1. When proving Theorem 1, we solve the problem of the representation theory: description of inequivalent representations of the extended Poincaré group which are realized on the set of solutions of system of nonlinear PDE (7). But the representation space (i.e., the set of solutions of system (7)) is not a linear vector space, whereas in the standard representation theory it is always the case. This fact makes impossible a direct application of the methods of linear representations [2].

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